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APPLICATION OF IMPLICIT FUNCTIONS METHODS TO SOLUTION OF THE NONLINEAR VECTOR SPECTRAL PROBLEM

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The non-linear spectral problems with vector spectral parameter for the case of holomorphic operator-function are considered. Use of implicit functions methods permits to determine qualitative characteristics of existing spectrum, and also to build simple computing algorithms for finding its connected components.

Keywords: nonlinear spectral problem, holomorphic operator-function, implicit functions method.

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1. The nonlinear spectral problems arise in different branches of analysis, mathematical physics and their applications. The theory and methods of solving the problems with one-dimensional spectral parameter are investigated most completely (see, in particular, [1-7]).

The non-linear spectral problems with vector n-dimensional parameter are considered in the work. Application of implicit functions methods to solution of this class of problems simplifies study of the qualitative characteristics of holomorphic operator-functions spectrum, and also it allows to formulate rather simple algorithms for numerical finding the connected components of spectrum.

2. Let H be a Hilbertian complex space, $\boldsymbol{\Lambda} = \Lambda_1 \times \Lambda_2 \times ... \times \Lambda_n$ – an open connected set in complex space $\mathbb{C}^n$ having elements of form $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_n)$, where $\lambda_i \in \Lambda_i \subset \mathbb{C} \quad (i = 1,\ 2,\ ...,\ n)$. The operator-function

$$A(\boldsymbol{\lambda}) = T(\boldsymbol{\lambda}) - I$$

is given, where $T(\boldsymbol{\lambda})$ is a linear continuous operator acting in the space H and analytically dependent on the vector parameter $\boldsymbol{\lambda}$, I is identical in H operator. Thus, the operator $A(\boldsymbol{\lambda}) \in \mathcal{L}(H, H)$ assigns to each value $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$.

We shall consider the non-linear spectral problem

$$A(\boldsymbol{\lambda})x = 0, \tag{1}$$

where it is necessary to find the eigenvalues $\boldsymbol{\lambda}^{(0)} = \left(\lambda_1^{(0)}, \lambda_2^{(0)}, ..., \lambda_n^{(0)}\right) \in \boldsymbol{\Lambda}$ and them eigenvectors corresponding to $x^{(0)} \in H$ $(x^{(0)} \neq 0)$ such that $A\left(\boldsymbol{\lambda}^{(0)}\right) x^{(0)} = 0$.

We shall choose in H an arbitrary total orthonormal system of functions $\{x_k\}_{k=1}^{\infty}$. Each element $x \in H$ can be written as a series $\sum\limits_{k=1}^{\infty} c_k x_k$, where $c_k = (x, x_k)$ is the Fourier coefficient of element x. Since $T(\boldsymbol{\lambda})$ is a linear continuous operator acting in the separable Hilbert space, it can be represented by matrices [8]

$$T_M(\boldsymbol{\lambda}) = \|t_{jk}(\boldsymbol{\lambda})\|_{j,k=0}^{\infty}, \tag{2}$$

where

$$t_{jk}(\boldsymbol{\lambda}) = (T(\boldsymbol{\lambda})x_j, x_k). \tag{3}$$

Thus the sequence of Fourier coefficients of element $y = T(\boldsymbol{\lambda})x$ is obtained from a sequence of Fourier coefficients of element x by transformation by matrix $T_M(\boldsymbol{\lambda})$.

Using the matrix formulation of operator $T_M(\boldsymbol{\lambda})$ the spectral problem (1) can be formulated as

$$A_M(\boldsymbol{\lambda})x \equiv (T_M(\boldsymbol{\lambda}) - I_M)\, x = 0 \tag{4}$$

where I_M is a unit matrix in the space of sequences l_2. Thus, operators $T(\boldsymbol{\lambda})$ and $T_M(\boldsymbol{\lambda})$ are equivalent in the sense that they assign to the same element $x \in H$ the same element $y \in H$. But we obtain the Fourier coefficients of element $y = T(\boldsymbol{\lambda})x$ as a result of action of operator $T_M(\boldsymbol{\lambda})$ on element x. Obviously, the spectrums of these operators will coincide, i.e. the spectral problems (1) and (4) are equivalent.

Let's consider function

$$F(\boldsymbol{\lambda}) = \det(T_M(\boldsymbol{\lambda}) - I_M)$$

which it is the determinant of finite dimension for the case when $T(\boldsymbol{\lambda})$ is degenerated operator [9] or space H is finite-dimensional. If $T(\boldsymbol{\lambda})$ is a non-degenerated linear continuous operator acting in the functional Hilbertian space H, the function $F(\boldsymbol{\lambda})$ we shall determine as

$$F(\boldsymbol{\lambda}) = \det \begin{pmatrix} t_{11}(\boldsymbol{\lambda}) - 1 & t_{12}(\boldsymbol{\lambda}) & ... & t_{1k}(\boldsymbol{\lambda}) & ... \\ t_{21}(\boldsymbol{\lambda}) & t_{22}(\boldsymbol{\lambda}) - 1 & ... & t_{2k}(\boldsymbol{\lambda}) & ... \\ .... \\ t_{k1}(\boldsymbol{\lambda}) & t_{k2}(\boldsymbol{\lambda}) & ... & t_{kk}(\boldsymbol{\lambda}) - 1 & ... \\ .... \end{pmatrix}.$$

Here we suppose that for $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ has place the inequality $|F(\boldsymbol{\lambda})| \leqslant C$, where C is some real constant.

Note, if $T(\boldsymbol{\lambda})$ is a completely continuous operator in H, then according to the definition of completely continuous operator [9], $T(\boldsymbol{\lambda})$ can be written as $T(\boldsymbol{\lambda}) = \tilde{T}_M(\boldsymbol{\lambda}) + T_\varepsilon(\boldsymbol{\lambda})$, where $\tilde{T}_M(\boldsymbol{\lambda})$ is a degenerated operator and the norm of operator $T_\varepsilon(\boldsymbol{\lambda})$ it is possible to make less than any preassigned small number $\varepsilon > 0$. Proceeding from this assumption, $T(\boldsymbol{\lambda})$ can be with prescribed accuracy approximated by the degenerated operator $\tilde{T}_M(\boldsymbol{\lambda})$ acting in the finite-dimensional space.

Obviously, $\boldsymbol{\lambda}^{(0)} \in \boldsymbol{\Lambda}$ will eigenvalue of the problem (1), if point $\boldsymbol{\lambda}^{(0)}$ is the solution to equation

$$F(\boldsymbol{\lambda}) = 0. \tag{5}$$

Together with this we shall consider also additional one-parameter spectral problem, corresponding to (1),

$$\tilde{A}(\lambda_1)x \equiv A(\lambda_1, f_1(\lambda_1), ..., f_n(\lambda_1))\, x = 0, \tag{6}$$

in which we suppose that $\lambda_i = f_i(\lambda_1)$ $(i = 2, ..., n)$, where $f_i(\lambda_1)$ are some single-valued differentiable functions mapping the domain Λ_1 into the domains $\Lambda_i' \subset \Lambda_i$ $(i = 2, ..., n)$. For the case of real parameters λ_i $(i = 1, 2, ..., n)$ functions $f_i(\lambda_1)$ $(i = 2, ..., n)$ determine some smooth curve belonging to $\boldsymbol{\Lambda}$.

Obviously, $\tilde{A}(\lambda_1) \in \mathcal{L}(E, E)$ at $\lambda_1 \in \Lambda_1$ and it is the narrowing of operator $A(\boldsymbol{\lambda}) \in \mathcal{L}(E, E)$. We shall designate the spectrum of operator-function $\tilde{A}(\lambda_1)$ by $s(\tilde{A})$. For the spectrum $s(A)$ of problems (1) takes place

Theorem. *Let for every* $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ the *operator* $A(\boldsymbol{\lambda}) \in \mathcal{L}(E, E)$ *be the Fredholm with zero index and operator-function* $A(\cdot) : \boldsymbol{\Lambda} \to\in \mathcal{L}(E, E)$ *be holomorphic, function* $F(\boldsymbol{\lambda})$ *be differentiable in the domain* $\boldsymbol{\Lambda}$. *Let also* $s(\tilde{A}) \neq \Lambda_1$.

Then:

1) Each point of spectrum $\lambda_1^{(0)} \in s(\tilde{A})$ *is isolated, it is an eigenvalue of operator* $\tilde{A}(\lambda_1) \equiv A(\lambda_1, f_1(\lambda_1), ..., f_n(\lambda_1))$, *the finite-dimensional eigen subspace* $N\left(\tilde{A}\left(\lambda_1^{(0)}\right)\right)$ *and finite-dimensional root subspace correspond to it;*

2) Each point $\boldsymbol{\lambda}^{(0)} = \left(\lambda_1^{(0)}, f_2(\lambda_1), ..., f_n(\lambda_1)\right) = \left(\lambda_1^{(0)}, \lambda_2^{(0)}, ..., \lambda_n^{(0)}\right) \in \boldsymbol{\Lambda}$ *is a point of operator spectrum* $A(\boldsymbol{\lambda})$;

3) If for some k $(1 \leqslant k \leqslant n)$ *the partial derivative* $F'_{\lambda_k}\left(\lambda_1^{(0)}, \lambda_2^{(0)}, ..., \lambda_n^{(0)}\right) \neq 0$, *then only one continuous differentiable function* $\lambda_k = \varphi(\lambda_1, ..., \lambda_{k-1}, \lambda_{k+1}, ...\lambda_n)$ *being the solution to equation (5) exists in some vicinity of point* $\left(\lambda_1^{(0)}, \lambda_2^{(0)}, ..., \lambda_n^{(0)}\right) \in \Lambda_1$, *that is in some polycircle area* $\boldsymbol{\Lambda}_0 = \left\{(\boldsymbol{\lambda}) \in \boldsymbol{\Lambda}_0 : \left|\lambda_i - \lambda_i^{(0)}\right| < \varepsilon_i\right\}$ *(ε_i are some real constants) there is a connected component of spectrum of operator-function* $A(\boldsymbol{\lambda})$.

□ *Proof.* Proof of the theorem follows from theorem 1 [4, p. 68] and implicit function existence theorem (see, for example, [10, 11]). At the beginning we shall show, that conditions of the Theorem 1 [4, p. 68][1], concerning the existence of discrete spectrum of operator-function $\tilde{A}(\lambda_1)$, follow from the conditions of formulated theorem. Since according to the conditions of theorem for every $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ $A(\boldsymbol{\lambda})$ is the Fredholm with zero index, and operator-function $A(\cdot) : \boldsymbol{\Lambda} \to\in \mathcal{L}(E, E)$ is holomorphic, then from this it follows, that at everyone $\lambda_1^{(0)} \in \Lambda_1$ operator $\tilde{A}(\lambda_1)$ is also Fredholm operator with zero index, and operator - function $\tilde{A}(\lambda_1) : \Lambda \to \mathcal{L}(E, E)$ is holomorphic. So, from the Theorem 1 [4, p. 68] it follows, that each point of spectrum $\lambda_1^{(0)} \in s(\tilde{A})$ is isolated. It is the eigenvalue of operator $\tilde{A}(\lambda_1)$ and the finite-dimensional eigen space and finite-dimensional root subspace correspond to it. The operator-function $\tilde{A}^{-1}(\lambda_1)$ has a pole at the point $\lambda_1^{(0)}$, which order is equal to the greatest length of root series, corresponding to $\lambda_1^{(0)}$ [2]. From this it follows, that each point $\boldsymbol{\lambda}^{(0)} = \left(\lambda_1^{(0)}, \lambda_2^{(0)}, ..., \lambda_n^{(0)}\right) = \left(\lambda_1^{(0)}, f_2\left(\lambda_1^{(0)}\right), ..., f_n\left(\lambda_1^{(0)}\right)\right) \in \boldsymbol{\Lambda}$ is eigenvalue of the operator $A(\lambda_1, f_1(\lambda_1), ..., f_n(\lambda_1)) \equiv \tilde{A}(\lambda_1)$. Thus, we have

$$F\left(\lambda_1^{(0)}, f_1\left(\lambda_1^{(0)}\right), ..., f_n\left(\lambda_1^{(0)}\right)\right) \equiv$$

$$\equiv F\left(\lambda_1^{(0)}, \lambda_2^{(0)}, ..., \lambda_n^{(0)}\right) = 0.$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be independent variables in domain $\boldsymbol{\Lambda}$, and $\left(\lambda_1^{(0)}, \lambda_2^{(0)}, ..., \lambda_n^{(0)}\right) \in \boldsymbol{\Lambda}$ is a point of spectrum of the operator $A(\boldsymbol{\lambda})$. Since $F(\boldsymbol{\lambda})$ is a differentiable function in the vicinity of point $\left(\lambda_1^{(0)}, \lambda_2^{(0)}, ..., \lambda_n^{(0)}\right)$ and $F'_{\lambda_k}\left(\lambda_1^{(0)}, \lambda_2^{(0)}, ..., \lambda_n^{(0)}\right) \neq 0$, then according to the implicit function theorem [11] in some vicinity of the point $\left(\lambda_1^{(0)}, \lambda_2^{(0)}, ..., \lambda_n^{(0)}\right)$ there exists a continuously differentiable function $\lambda_k = \varphi(\lambda_1, ..., \lambda_{k-1}, \lambda_{k+1}, ...\lambda_n)$ being the solution to equation (4). From this it follows that the connected components of operator-function of spectrum $A(\cdot) : \boldsymbol{\Lambda} \to\in \mathcal{L}(E, E)$ exist in some polycircle domain $\boldsymbol{\Lambda}_0 = \left\{(\boldsymbol{\lambda}) \in \boldsymbol{\Lambda}_0 : \left|\lambda_i - \lambda_i^{(0)}\right| < \varepsilon_i\right\}$.

Theorem is proved. ■

3. As an example we shall consider a particular case of the problem (1) supposing that $T_M(\boldsymbol{\lambda})$ is a square λ-matrix [12] with respect to three variables of order m, i.e. we shall consider, that its coefficients are polynomials

[1] Note, proof of the theorem 1 [4, p. 68] is based on results of works [2,3].

$$t_{ij}(\boldsymbol{\lambda}) = P_{ij}^{(k_{ij},n_{ij},m_{ij})}(\lambda_1,\lambda_2,\lambda_3) \equiv$$

$$\equiv a_{ij}^{(0,0,0)} + a_{ij}^{(1,0,0)}\lambda_1 + a_{ij}^{(0,1,0)}\lambda_2 + a_{ij}^{(0,0,1)}\lambda_3 + ... + \quad (7)$$
$$+ a_{ij}^{(k_{ij},n_{ij},m_{ij})}\lambda_1^{k_{ij}}\lambda_2^{m_{ij}}\lambda_3^{n_{ij}},$$

where $a_{ij}^{(k_{ij},n_{ij},m_{ij})}$ are real or complex coefficients. Let's assume, that $k_{ij} + n_{ij} + m_{ij} \leqslant \alpha$ for all $i, j = 1, 2, ..., n$. Obviously, that the operator-function $A_M(\boldsymbol{\lambda}) = T_M(\boldsymbol{\lambda}) - I_M$ is holomorphic at any point $\boldsymbol{\lambda}^{(0)} \in \boldsymbol{\Lambda}$, where $\lambda_i \in \Lambda_i = \mathbb{C}$. Since for any fixed value $\boldsymbol{\lambda}^{(0)} \in \boldsymbol{\Lambda}$ the equality [12]

$$\dim\left(\ker T_M\left(\boldsymbol{\lambda}^{(0)}\right)\right) = \dim\left(\ker T_M^*\left(\boldsymbol{\lambda}^{(0)}\right)\right)$$

is true then for each $\boldsymbol{\lambda}^{(0)} \in \boldsymbol{\Lambda}$ the operator-function $A_M(\boldsymbol{\lambda}^{(0)})$ is the Fredholm operator with zero index operating in the space $\mathbb{C}^m$.

Let's assume, in particular, in $T_M(\lambda_1,\lambda_2,\lambda_3)$, $\lambda_2{=}\lambda_1$, $\lambda_3 = \lambda_1$ where $\lambda_1 \in \Lambda_1$. Thus, coefficients of matrix $t_{ij}(\lambda)$ become

$$\tilde{t}_{ij}(\lambda_1) = \tilde{P}_{ij}^{(k_{ij},m_{ij},n_{ij})} =$$
$$= a_{ij}^{(0,0,0)} + \left(a_{ij}^{(1,0,0)} + a_{ij}^{(0,1,0)} + a_{ij}^{(0,0,1)}\right)\lambda_1 + ... +$$
$$+ \left(a_{ij}^{(k_{ij},0,0)} + a_{ij}^{(0,m_{ij},0)} + a_{ij}^{(0,0,n_{ij})}\right)\lambda_1^{k_{ij}+m_{ij}+n_{ij}}$$

and function $\tilde{F}(\lambda_1){=}\det(T_M(\lambda_1){-}I_M){=}0$ is a polynomial which degree does not exceed $\beta = \alpha m$. So, according to the basic theorem of algebra of polynomials the equation $\tilde{F}(\lambda_1) = 0$ has no more than β roots. From this follows the existence of solutions to the auxiliary problem (6). A set of elements $\left\{\lambda_1^{(\nu)}, \lambda_2^{(\nu)}{=}\lambda_1^{(\nu)}, \lambda_3^{(\nu)}{=}\lambda_1^{(\nu)}\right\}{\in}\Lambda_1{\times}\Lambda_2{\times}\Lambda_3$ corresponding to the roots of this equation $\lambda_1^{(\nu)} \in \Lambda_1$ will be a discrete set of eigenvalues of operator $A_M(\lambda_1,\lambda_2,\lambda_3)$ belonging to the beam $\{\lambda_2 = \lambda_1,\ \lambda_3 = \lambda_1,\ \lambda_1 \in \Lambda_1\}$.

Consider equation (5) supposing that coefficients of matrix $T_M(\lambda_1,\lambda_2,\lambda_3)$ are determined by formulas (7). The existence and continuity of partial derivatives $\frac{\partial F(\lambda_1,\lambda_1,\lambda_3)}{\partial\lambda_k}$, $k = 1, 2, 3$, is obvious. If for ν -th point $\left\{\lambda_1^{(\nu)}, \lambda_2^{(\nu)}, \lambda_3^{(\nu)}\right\}$ at least one of partial derivatives $\frac{\partial F(\lambda_1^{(\nu)},\lambda_2^{(\nu)},\lambda_3^{(\nu)})}{\partial\lambda_k}$ is distinct from zero (for example, for $k = 2$), then according to the implicit functions theorem [10] in some vicinity of ν-th point $\left\{\lambda_1^{(\nu)}, \lambda_2^{(\nu)}, \lambda_3^{(\nu)}\right\}$ there exists only one differentiable function $\lambda_2^{(v)}(\lambda_1,\lambda_3)$, describing the ν-th connected component of spectrum of λ-matrix $T_M(\boldsymbol{\lambda})$. Function $\lambda_2^{(v)}(\lambda_1,\lambda_3)$ can be found, in particular, by successive approximations method proposed in [10].

The points $\left\{\lambda_1^{(\nu)}, \lambda_2^{(\nu)}, \lambda_3^{(\nu)}\right\}$, at which all partial derivatives $\frac{\partial F(\lambda_1^{(\nu)},\lambda_2^{(\nu)},\lambda_3^{(\nu)})}{\partial\lambda_k} = 0$, $k = 1, 2, 3$, are special. In detail they are investigated in [10].

4. For the case of two-dimensional non-linear spectral problem finding the connected components of spectrum as solutions of equation $F(\lambda_1,\lambda_2) = 0$, is reduced to solution of the Caushy problem for the first order differential equation. If at points $\left(\lambda_1^{(\nu)}, \lambda_2^{(\nu)}\right) \in \boldsymbol{\Lambda}$ the derivative $F'_{\lambda_2}(\lambda_1,\lambda_2)$ is distinct from zero, we find the ν-th connected component of spectrum of the operator $A_M(\lambda_1,\lambda_2) = T_M(\lambda_1,\lambda_2) - I_M$ solving the Caushy problem

$$\frac{d\lambda_2}{d\lambda_1} = -\frac{F'_{\lambda_1}(\lambda_1,\lambda_2)}{F'_{\lambda_2}(\lambda_1,\lambda_2)}, \quad \lambda_2\left(\lambda_1^{(\nu)}\right) = \lambda_2^{(\nu)}$$

in the vicinity of each point $\lambda_1^{(\nu)} \in \Lambda_1$.

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ЗАСТОСУВАННЯ МЕТОДІВ НЕЯВНИХ ФУНКЦІЙ ДО РОЗВ'ЯЗАННЯ НЕЛІНІЙНОЇ ВЕКТОРНОЇ СПЕКТРАЛЬНОЇ ПРОБЛЕМИ

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Розглядаються нелінійні спектральні задачі з векторним спектральним параметром для випадку голоморфних оператор-функцій. Застосування методів неявних функцій дає змогу визначати якісні характеристики існуючого спектра, а також будувати нескладні обчислювальні алгоритми для знаходження його зв'язних компонент.

Ключові слова: нелінійна спектральна проблема, голоморфна оператор-функція, метод неявних функцій.

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