On The Hyperbolic Simulations to the Camassa-Holm Equation

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Abstract—In this paper, we apply the sine-Gordon expansion method to the Camassa-Holm equation. We obtain some new travelling wave solutions such as complex, hyperbolic and trigonometric function solutions. All the travelling wave solutions are verified by using Wolfram Mathematica 9 and they are indeed solutions to the model. We also plot the two- and three-dimensional surfaces for all the travelling wave solutions obtained in this paper using the same computer program.

Key words—The sine-Gordon expansion method; the Camassa-Holm equation; complex function solution; hyperbolic function solution; trigonometric function solution.

I. Introduction

The studies of the solution structures to the nonlinear evolution equations (NLEEs) have become of significance important in the fields of science, mathematical physics and engineering, chemistry, plasma physics, fluid mechanics etc. Various efforts from different researchers have been implemented to investigate the solutions to such class of equations such as improved Bernoulli sub-equation function method [1], modified exp(−ϕ(ξ)) -expansion method [2], (G'/G)-expansion method [3], exp-function method [4], Sumudu transform method [5], tanh-coth method [6] etc.

In this study, we employ sine-Gordon expansion method (SGEM) to the well known NLEE, namely; the Camassa-Holm (CH) equation [7] given as following

\[ u_t + 2\alpha u_x - u_{xx} + buu_x = 0, \]  

(1.1)

where \( \alpha, b \) are real constants. Camassa-Holm equation in Eqn. (1.2) derived by Camassa and Holm is an integrable nonlinear partial differential equation that describes the model of a shallow water waves [8]. Later, an effort has been employed by [9] and obtained the solution of Eqn. (1.1) in its lowest order by making the right hand side of Eqn. (1.2) to be small (negligible)

\[ u_t + 2\alpha u_x - u_{xx} + buu_x = 2u_x u_{xx} + uu_{xxx}. \]  

(1.2)

Several attempts have been made to investigate the nature of the solution to CH equation, this includes [10-14] The remaining part of this paper is organized as follows: In Section 2, we give a description of the SGEM. In Section 3, we implement the SGEM to Eqns. (1.1). We finally, give the conclusion of this study in section 4.

II. General Facts of SGEM

In this section of paper, we give a description of sine-Gordon expansion method on how it finds a new travelling wave solutions to the nonlinear evolution equations.

Sine-Gordon expansion method is based on sine-Gordon equation and travelling wave transformation [15,16]. Lets consider the following equation

\[ u_{xx} - u_t = m^2 \sin(u), \]  

(2.1)

where \( u = u(x,t) \), and \( m \) is real constant. When we apply the wave transform \( \xi = \mu(x - ct) \) to Eq.(2.1), we obtain the nonlinear ordinary differential equation (NODE) as following;

\[ U'' = \frac{m^2}{\mu^2(1-c^2)} \sin(U), \]  

(2.2)

where \( U = U(\xi) \), and, \( \xi \) is the amplitude of the travelling wave, \( c \) is the velocity of the travelling wave. If we reconsider Eq.(2.2), we can write in the fullsimplified version as following:

\[ \left( \frac{U}{2} \right)^2 = \frac{m^2}{\mu^2(1-c^2)} \sin^2\left( \frac{U}{2} \right) + K, \]  

(2.3)

where \( K \) is the integration constant. When we resubmit as \( K = 0, w(\xi) = \frac{U}{2}, \) and \( \alpha^2 = \frac{m^2}{\mu^2(1-c^2)} \) in Eq.(2.3), we can obtain following equation;

\[ w' = a \sin(w). \]  

(2.4)

If we put as \( a = 1 \) in Eq.(2.4), we can obtain following equation;

\[ w' = \sin(w). \]  

(2.5)

If we solve Eq.(2.5) by using separation of variables, we find the following two significant equations;

\[ \sin(w) = \sin(w(\xi)) = \frac{2p_{1^{\pm}}}{p^2 e^{2p_{1^{\pm}}} + 1}, \]  

(2.6)

or

\[ \cos(w) = \cos(w(\xi)) = \frac{p^2 e^{2p_{1^{\pm}}} - 1}{p^2 e^{2p_{1^{\pm}}} + 1}, \]  

(2.7)

where \( p \) is the integral constant and non-zero. For obtaining the solution of following nonlinear partial differential equation;

\[ P(u, u_x, u_{xx}, \cdots) \]  

(2.8)
let’s consider the travelling wave solution as

$$U(\xi) = \sum_{i=1}^{n} \tanh^{i-1}(\xi)[B_i \sec h(\xi)] + A_i \tanh(\xi)] + A_0.$$  \hspace{1cm} (2.9)

We can rewrite Eq.(2.9) according to Eqs.(2.6) and Eq.(2.7) as following:

$$U(w) = \sum_{i=1}^{n} \cos^{i-1}(w)[B_i \sin(w)] + A_i \cos(w)] + A_0.$$ \hspace{1cm} (2.10)

Under the terms of homogenous balance technique, we can determine the values of \(n\) under the terms of NODE. Let the coefficients of \(\sin'(w)\cos'(w)\) all be zero, it yields a system of equations. Solving this system by using Wolfram Mathematica 9 give the values of \(A_i, B_i, \mu, c\). Finally, substituting the values of \(A_i, B_i, \mu, c\) in Eq.(2.10), we can find the new travelling wavesolutions to the Eq.(2.8).

III. Implementation of SGEM

In this section we consider solving Eqn. (1.1) by using SGEM. Consider the Camassa-Holm equation given in Eqn. (1.1). Using the transformation \(u = U(\xi)\), \(\xi = \mu(x - ct)\), Eqn. (1.1) reduces to the following NODE:

\[(4\alpha - c)U + bU^2 + 2c\mu^2U'' = 0,\]  \hspace{1cm} (3.1)

using the balance principle on the highest derivative \(U''\) and the nonlinear term \(U^2\) in Enq. (3.1), yields \(n = 2\). Using \(n = 2\) along with Eqn. (2.10), we have:

$$U(w) = B_1 \sin(w) + A_1 \cos(w) + A_0 + B_2 \cos(w)\sin(w) + A_2 \cos^2(w),$$ \hspace{1cm} (3.2)

$$U'(w),$$

$$U''(w),$$

$$\vdots$$

substituting Eqns. (3.2) and (3.3) in Eqn. (3.1), we have a system including some trigonometric functions. We obtain an algebraic system of these functions by equating all the coefficients of the trigonometric identities of the same power to zero. We solve the system of algebraic equations with help of Wolfram Mathematica 9 and substitute in each case the obtained results of the coefficients in Eqn. (2.9), to obtain the new travelling solutions to equation (1.1).
Case-4:

\[ A_0 = \frac{2\kappa \bar{\mu}}{b(1 - 4\mu^2)}; A_1 = 0, B_1 = 0, A_2 = \frac{2\kappa \mu}{b(1 - 4\mu^2)}, B_2 = 0, \alpha = \frac{2\kappa}{(1 - 4\mu^2)}. \]

\[ u_0(x, t) = \frac{2\kappa \mu \text{sech}^2[\alpha(x - ct)]}{b(1 - 4\mu^2)}. \]

(3.7)

Figure 4: The 3D and 2D surfaces of Eqn. (3.7) by considering the values \( \alpha = 1, \mu = 0.5, b = 2, -5 < x < 5, -2 < t < 2 \) and \( t = 0.01 \) for the 2D graph.

Case-5:

\[ A_0 = \frac{2\kappa \mu}{b}, A_1 = 0, B_1 = 0, A_2 = \frac{2\kappa \mu}{b}, B_2 = 0, \alpha = \frac{1}{2}(1 + 4\mu^2). \]

\[ u_0(x, t) = \frac{4\kappa \mu(1 - 3\text{tanh}^2[\mu(x - ct)])}{b}. \]

(3.8)

Figure 5: The 3D and 2D surfaces of Eqn. (3.8) by considering the values \( \mu = 0.5, b = 2, -5 < x < 5, -2 < t < 2 \) and \( t = 0.01 \) for the 2D graph.

Case-6:

\[ A_0 = \frac{2\kappa \mu}{b}, A_1 = 0, B_1 = 0, A_2 = -\frac{6\kappa \mu}{b}, B_2 = -\frac{6\kappa \mu}{b}, \alpha = \frac{1}{2}(1 - \mu^2). \]

\[ u_0(x, t) = \frac{6\kappa \mu \text{sech}^2[\mu(x - ct)](\text{sech}^2[\mu(x - ct)] - \text{tanh}^2[\mu(x - ct)])}{b}. \]

(3.9)

Figure 6: The 3D and 2D surfaces of Eqn. (3.9) by considering the values \( \mu = 0.5, b = 2, -5 < x < 5, -2 < t < 2 \) and \( t = 0.01 \) for the 2D graphs.

Case-7:

\[ A_0 = \frac{6\kappa \mu^2}{b}, A_1 = 0, B_1 = 0, A_2 = -\frac{6\kappa \mu^2}{b}, B_2 = -\frac{6\kappa \mu^2}{b}, \]

\[ \mu = -\frac{6\kappa \mu^2}{b}. \]

\[ u_0(x, t) = \frac{6\kappa \mu^2 \text{sech}^2[\mu(x - ct)](\text{sech}^2[\mu(x - ct)])}{b}. \]

(3.10)

Figure 7: The 3D and 2D surfaces of Eqn. (3.10) by considering the values \( \mu = 0.5, b = 2, -5 < x < 5, -2 < t < 2 \) and \( t = 0.01 \) for the 2D graph.

Case-8:

\[ A_0 = \frac{3\kappa \mu}{b}, A_1 = 0, B_1 = 0, A_2 = \frac{3\kappa \mu}{b}, B_2 = 0, \alpha = \frac{\sqrt{3} - 2}{\sqrt{3} + 2}. \]

\[ u_0(x, t) = \frac{3\kappa \mu \text{sech}^2[\mu(x - ct)]}{b}. \]

(3.11)

Figure 8: The 3D and 2D surfaces of Eqn. (3.11) by considering the values \( \mu = 0.5, b = 2, -5 < x < 5, -2 < t < 2 \) and \( t = 0.01 \) for the 2D graph.

Case-9:

\[ A_0 = \frac{6\kappa \mu}{b}, A_1 = 0, B_1 = 0, A_2 = \frac{6\kappa \mu}{b}, B_2 = 0, \alpha = 4\kappa \mu, \mu = \frac{1}{2\sqrt{2}}. \]

\[ u_0(x, t) = \frac{6\kappa \mu \text{sech}^2[\mu(x - ct)]}{b}. \]

(3.12)

Figure 9: The 3D and 2D surfaces of Eqn. (3.12) by considering the values \( \mu = 0.5, b = 2, -5 < x < 5, -2 < t < 2 \) and \( t = 0.01 \) for the 2D graph.

Case-10:

\[ A_0 = \frac{6\kappa \mu}{b}, A_1 = 0, B_1 = 0, A_2 = \frac{6\kappa \mu}{b}, B_2 = -\frac{6\kappa \mu}{b}, \alpha = -\frac{6\kappa \mu}{b}. \]

\[ u_0(x, t) = \frac{6\kappa \mu \text{sech}^2[\mu(x - ct)]}{b}. \]

(3.13)

Figure 10: The 3D and 2D surfaces of Eqn. (3.13) by considering the values \( \mu = 0.5, b = 2, -5 < x < 5, -2 < t < 2 \) and \( t = 0.01 \) for the 2D graph.
Case-11:

In this paper, we have applied the sine-Gordon expansion method to the Camassa-Holm equation. We obtained some new travelling wave solutions such as complex function solutions, trigonometric function solutions and hyperbolic function solutions to equation (1.1). The obtained travelling wave solutions to this equation are indeed verified to be the solutions to the equation with help of Wolfram Mathematica 9. Thus, to obtain such hyperbolic solutions for nonlinear evolution equations in shallow water wave model is very important to know the physical meaning of the solutions.

For instance, the hyperbolic sine arises in the gravitational potential of cylinder, the hyperbolic cosine function is the shape of a hanging cable and the hyperbolic tangent arises in the calculation and rapidity of special relativity [18].

We think that some results found in this paper are related to such physica meaning. We observed that the solutions to equation (1.1) are newly constructed solutions when compared with the solutions obtained by [7].

We can finally say that sine-Gordon expansion method is a powerful tool that can be applied to various nonlinear evolution equations.

To the best of our knowledge, the application of SGEM to (1.1) and has not been submitted to literature in advance.

References