Exponentially Fitted Methods on Layer-Adapted Mesh for Singularly Perturbed Delay Differential Equations

Fevzi Erdogan
Yuzuncu Yil University, Faculty of Sciences, Department of Mathematics, Van, TURKEY, E-mail: fevzier@gmail.com

Abstract—The purpose of this study is to present a uniform finite difference method for numerical solution of a initial value problem for quasi-linear second order singularly perturbed delay differential equation. A numerical method is constructed for this problem which involves appropriate piecewise-uniform Shishkin mesh on each time subinterval. The method is shown to uniformly converge with respect to the perturbation parameter. A numerical experiment illustrate in practice the result of convergence proved theoretically.

Key words — The finite difference method, Appropriate piecewise-uniform Shishkin mesh.

I. Introduction

Consider an initial value problem for the linear second order singularly perturbed delay differential equation

\[ \varepsilon u'(t) + a(t)u(t) + f(t, u(t), u(t-r)) = 0, \quad t \in I, \]

\[ u(t) = \varphi(t), \quad t \in I_0, \]

\[ u'(0) = A / \varepsilon, \]  \tag{3}

where \( I = (0, T), I_0 = (-r, 0], 0 < \varepsilon \leq 1 \) is the perturbation parameter, \( a(t) \geq \alpha > 0 \), \( f(t) \) and \( \varphi(t) \) are given sufficiently smooth functions satisfying certain regularity conditions to be specified and \( r \) is a constant delay.

Delay differential equations play an important role in the mathematical modelling of various practical phenomena in the biosciences and control theory. Any system involving a feedback control will almost always involve time delays. These arise because a finite time is required to sense information and then react to it. A singularly perturbed delay differential equation is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and \( r \) is a constant delay.

In a singularly perturbed delay differential equation, one encounters with two difficulties, one is because of occurrence of the delay term and another one is due to presence of perturbation parameter. To overcome the first difficulty, we employed the numerical method of steps [2] for the delay argument which converted the problem to a initial value problem for a singularly perturbed differential equation. Then we constructed a numerical scheme based on finite difference method on non-uniform Shishkin mesh for the numerical solution.

In the present paper we discretize the problem(1)-(3) using a numerical method, which is composed of an exponentially fitted difference scheme on piecewise uniform Shishkin mesh on each time subinterval. In section 2, we state some important properties of the exact solution. In section 3, we describe the finite difference discretization and introduce the piecewise uniform mesh. In section 4, we present convergence analysis for approximate solution. Uniform convergence is proved in the discrete maximum norm. Some numerical results are being presented in section 5. The technique to construct discrete problem and error analysis for approximate solution is similar to those in [8,9,22,23].

II. Discretization and Mesh

In this section, we construct a numerical scheme for solving the initial value problem (1)-(3). We propose the following difference scheme for approximation (1)-(3).

\[ \varepsilon \theta_i \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + a_i \frac{y_{i+1} - y_{i-1}}{2h} + f(t_i, y_i, y_{i-r}) = 0, \]

\[ \theta_i = \varphi_i, \quad -N \leq i \leq 0, \quad \varepsilon \sigma(y_i - y_0) - A h = 0, \]  \tag{4}

where \( \theta_i \) and \( \sigma \) are defined by

\[ \theta_i = \frac{h a_i}{2 \varepsilon \coth(\frac{h a_i}{2 \varepsilon})}, \quad \sigma = \frac{h a_i}{2 \varepsilon} (1 - e^{-\frac{h a_i}{\varepsilon}}) \]

The difference scheme (1)-(3), in order to be \( \varepsilon \) -uniform convergent, will use the Shishkin mesh. For the even number \( N \), the piecewise uniform mesh \( \sigma_{p_0} \) divides each of the interval \( [r_{p-1}, \sigma_p] \) and \( [\sigma_p, r_p] \) into \( N/2 \) equidistant subintervals, where the transition point \( \sigma_p \), which separates the fine and coarse portions of the mesh is obtained by

\[ \sigma_p = r_{p-1} + \min \{ r / 2, \alpha^{-1} \varepsilon \ln N \}. \]
III. Convergence Analysis

We now estimate the approximate error \( z_i = y_i - u_i \), which satisfies the discrete problem

\[
\varepsilon \partial_t z_{i+1} - 2z_i + z_{i-1} + \frac{f(t_i, y_i, y_{i-N}) - f(t_i, u_i, u_{i-N})}{2h} = R_i, \quad i = 1, 2, ..., N,
\]

where \( R_i \) are the truncation errors. Then the following estimate holds

\[
|y_i - u_i| \leq CN^{-1} \ln N, \quad 0 \leq i \leq N.
\]

IV. Theorem

The continuously differentiable function \( f(t, u, v) \) satisfies the regularity conditions and the derivative \( \frac{\partial}{\partial t} f(t, u, v) \) is bounded for given interval. Then the following estimate holds

\[
\varepsilon \sigma(z_i - z_0) - Ah + r^0 = 0,
\]

where \( R_i \) and \( r^0 \) are the truncation errors.

V. Numerical Results

In this section, a simple numerical example is devised to verify the validity for the proposed method. Consider the test problem with

\[
a = 1, f = -u(t-1) + \dot{r}^2, T = 2, \varphi = 1 + t, A = -1.
\]

We use the double mesh principle to estimate the errors and compute the experimental rates of convergence in our computed solution, i.e. We compare the computed solution with the solution on a mesh that is twice as fine [17].

Maximum errors and rates of convergence on \( \omega_{N,1} \) and \( \omega_{N,2} \)

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>N=64</th>
<th>N=128</th>
<th>N=256</th>
<th>N=512</th>
<th>N=1024</th>
</tr>
</thead>
<tbody>
<tr>
<td>2^{-1}</td>
<td>0.0079814</td>
<td>0.0040103</td>
<td>0.0020101</td>
<td>0.0010063</td>
<td>0.0005033</td>
</tr>
<tr>
<td>2^{-8}</td>
<td>0.0191427</td>
<td>0.0114094</td>
<td>0.0063031</td>
<td>0.0037425</td>
<td>0.0010063</td>
</tr>
<tr>
<td>2^{-16}</td>
<td>0.0190685</td>
<td>0.0113652</td>
<td>0.0065754</td>
<td>0.0037820</td>
<td>0.0010063</td>
</tr>
</tbody>
</table>

Conclusion

In this study we have presented a numerical approach to solve a semi-linear singularly perturbed first-order delay differential equation, using an exponentially fitting factor for the difference scheme. We proposed an exponentially fitted difference scheme on piecewise-uniform Shishkin mesh on each time subinterval. We have shown that the method displays uniform convergence with respect to the perturbation parameter for numerical approximation of the solution.

References