Travelling Wave Simulations to the Modified Zakharov-Kuzentsov Model Arising In Plasma Physics

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Abstract — In this manuscript, we carry out the modified exp-(−Ω(xi))-expansion function method to the modified Zakharov-Kuzentsov equation with (2+1) dimensions arising in plasma physics. Then, the hyperbolic and complex travelling wave solutions are obtained to the model. It is observed that all results are verified to the model with the help of Wolfram Mathematica 9. We also plot the two- and three-dimensional surfaces for all the travelling wave solutions obtained in this paper using the same computer program.

Key words — The modified exp-(−Ω(xi))-expansion function method; the modified Zakharov-Kuzentsov equation; complex and hyperbolic function solution.

I. Introduction

To obtain new physical properties of differential equations arising in plasma physics is very important in the terms of the better understanding of physical meaning of models. That is why many experts and scientists are studying on this new concept. In the modern world, many actual problems are modelling by using mathematical models, especially, in the fields of science, mathematical physics and engineering, chemistry, plasma physics, fluid mechanics etc. These new models have required new methods which give many new physical properties of actual problems. Some of such methods which are powerful one for models such as sine-Gordon expansion method, improved Bernoulli sub-equation function method, He’s Semi-Inverse Method, the extended trial equation method etc [1-4] have submitted to the literature.

In this study, we apply modified exp-(−Ω(xi))-expansion function method (MEFM) to the modified Zakharov-Kuzentsov equation (ZK) with (2+1)-dimensions arising in plasma physics given as following

\[ u_t + \beta u^2 u_x + u_{xxx} + u_{yxx} = 0, \]  

where \( \beta \) is a real constant and non-zero. The modified ZK in (2+1) dimensions Eq.(1) is a mathematical model for acoustic plasma waves [5,6].

The remaining part of this paper is organized as follows: In Section 2, we give a description of the MEFM. In Section 3, we implement the MEFM to Eqn. (1). We finally, give the conclusion of this study in the section 4.

II. General Facts of MEFM

The general properties of modified exp-(−Ω(xi))-expansion function method (MEFM) are proposed in this section. MEFM is based on the exp-(−Ω(xi))-expansion function method [7-9]. In order to apply this method to the nonlinear partial differential equations, we consider the following equation:

\[ P(u, u_x, u_y, u_t, u_{xx}, u_{yy}, u_{tt}, \ldots) = 0, \]  

where, \( u = u(x, y, t) \) is an unknown function, \( P \) is a polynomial in \( u \). The basic fundamentals of MEFM are expressed as follows:

Step 1: Let’s consider the following travelling wave transformation defined as

\[ u(x, y, t) = U(\xi), \quad \xi = kx + py + rt, \]

where \( k, p, r \) are real constant and non zero. Using Eq.(3), we can convert Eq.(2) into the nonlinear ordinary differential equation (NODE) defined by;

\[ NODE(U, U', U'', U'''', \ldots) \]

where \( NODE \) is a polynomial of \( U \) and its derivatives and \( U' = dU/d\xi \) indicate the ordinary derivatives with respect to \( \xi \).

Step 2: Suppose the travelling wave solution of Eq.(4) can be rewritten as following manner;

\[ U(\xi) = \sum_{i=0}^{N} A_i e^{\alpha_i \xi} + \sum_{i=0}^{M} B_i e^{\beta_i \xi}, \]

where \( A_i, B_j, (0 \leq i \leq N, 0 \leq j \leq M) \) are constants to be determined later, such that \( A_i \neq 0, B_m \neq 0 \), and \( \Omega = \Omega(\xi) \) verify the following ordinary differential equation;

\[ \Omega'(\xi) = e^{\alpha(\xi) + \mu e^{\alpha(\xi) + \lambda}}. \]

Eq. (6) has the following solution families [7-9]:

Family-1: When \( \mu \neq 0, \lambda^2 - 4\mu > 0 \),

\[ \Omega(\xi) = \ln \left( \frac{-\sqrt{\lambda^2 - 4\mu} \tanh(\kappa)}{2\mu} - \frac{\lambda}{2\mu} \right), \]

where \( \kappa = \frac{\sqrt{\lambda^2 - 4\mu}}{2}(\xi + E) \).

Family-2: When \( \mu \neq 0, \lambda^2 - 4\mu < 0 \),

\[ \Omega(\xi) = \ln \left( \frac{-\sqrt{\lambda^2 - 4\mu} \tan(\alpha)}{2\mu} - \frac{\lambda}{2\mu} \right), \]

where \( \alpha = \frac{-\sqrt{\lambda^2 - 4\mu}}{2}(\xi + E) \).

Family-3: When \( \mu = 0, \lambda \neq 0, \) and \( \lambda^2 - 4\mu > 0 \),

\[ \Omega(\xi) = -\ln \left( \frac{\lambda}{\exp(\lambda(\xi + E)) - 1} \right). \]
Family-4: When \( \mu \neq 0, \lambda \neq 0, \) and \( \lambda^2 - 4\mu = 0, \)
\[
\Omega(\xi) = \ln \left( -\frac{2\lambda(\xi + E)}{\lambda^2 (\xi + E)} + 4 \right), \tag{10}
\]

Family-5: When \( \mu = 0, \lambda = 0, \) and \( \lambda^2 - 4\mu = 0, \)
\[
\Omega(\xi) = \ln (\xi + E), \tag{11}
\]
such that \( A_0, A_1, A_2, \ldots \) are constants to be determined later. The positive integers \( N \) and \( M \) can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms occurring in Eq. (5).

**Step 3:** Substituting Eq. (6) and Eq. (7) into Eq. (5), we get a polynomial of \( \exp\left(-\Omega(\xi)\right) \). We equate all the coefficients of same power of \( \exp\left(-\Omega(\xi)\right) \) to zero. This procedure yields a system of equations which can be solved to find \( A_0, A_1, A_2, \ldots, B_0, B_1, B_2, \ldots, l,\mu \) with the aid of commercial software programming Wolfram Mathematica 9. Substituting the values of \( A_0, A_1, A_2, \ldots, B_0, B_1, B_2, \ldots, l,\mu \) in Eq.(5), the general solutions of Eq. (5) complete the determination of the solution of Eq. (1).

### III. Implementation of MEFM

In this section we find the complex and hyperbolic travelling wave solutions to the Eq. (1) by using MEFM. Using the transformation \( \xi = kx + py + rt \) in the Eq(1), it reduces to the following NODE:

\[
3rU + \beta kU^3 + 3(k^2 + kp^2)U'' = 0. \tag{12}
\]

If we use the balance principle between the highest derivative \( U'' \) and the nonlinear term \( U^3 \) in Eq.(12), it yields

\[
N = M + 1. \tag{13}
\]

This relationship produces many travelling wave solutions to the model considered in the paper.

**Case-1:** When we take \( M = 1 \) and \( N = 2 \) in Eq.(5), we can write follows:

\[
U = A_0 + A_1 e^{\alpha x} + A_2 e^{2\alpha x} \tag{14}
\]

\[
U' = \left[ A_1 e^{\alpha x}(-2\Omega') + A_2 e^{2\alpha x}(-2\Omega) \right] \tag{15}
\]

\[
U'' = \frac{A_1 e^{\alpha x}(-\Omega') + A_2 e^{2\alpha x}(-2\Omega)}{B_0 + B_1 e^{\alpha x}} - \frac{A_1 + A_2 e^{\alpha x} + A_2 e^{2\alpha x} B_0 e^{\alpha x}(-\Omega')}{B_0 + B_1 e^{\alpha x}} = \frac{Y\Psi'}{\Psi^2}, \tag{16}
\]

\[
U^* = \frac{\Psi'^2 - \Psi Y''}{\Psi^2},
\]

where \( A_2 \neq 0 \) and \( B_1 \neq 0 \). Substituting Eqs.(14,15) in Eq.(12), we get an equation including \( e^{\alpha x} \) and its various powers. Therefore, we have a system of equations from the coefficients of polynomial of \( e^{\alpha x} \). Solving this system of equations yields the following coefficients:

\[
A_0 = \mu A_2, \quad A_1 = \lambda A_2, \quad B_0 = 0.5\lambda B_1,
\]

\[
p = \frac{-1}{\sqrt{6B_1}} \sqrt{-6k^2B_1^2 - \beta A_2^2}, \quad r = \frac{k\beta A_2^2}{6B_1^2} (\lambda^2 - 4\mu), \tag{16}
\]

Using coefficients of Eq.(16) along with Eqs.(3) and (7) into Eq. (14), we obtain a new hyperbolic function solution to the modified Zakharov-Kuzentsov equation (ZK) with \( (2+1) \)-dimensions arising in plasma physics as following;

\[
u_i = \frac{-2k\mu A_2 \sec h^2 f \left( f \right)}{B_i \left( \lambda + \sqrt{\kappa} \tanh \left( f \right) \right) \left( \lambda + \sqrt{\kappa} \tanh \left( f \right) \right)}, \tag{17}
\]

where \( \kappa = \lambda^2 - 4\mu \) and

\[
f = \frac{1}{2} \sqrt{\kappa \left( E + kx + \frac{k\beta}{6B_1^2} \right)}.
\]

\[
p = \frac{1}{\sqrt{6B_1}} \sqrt{-6k^2B_1^2 - \beta A_2^2} \quad \text{and} \quad \lambda^2 - 4\mu > 0 \]

with the under the terms of **Family-1**.

**Fig. 1.** The 3D and 2D surfaces of the travelling wave structure Eq.(17) being hyperbolic function solutions for \( \lambda = 4, \mu = 2, k = 0.3, A_2 = 3, B_1 = 5, \beta = -8, E = 6, y = 7, 0 < x < 20, 0 < t < 10 \) and \( t = 5 \) for 2D graphcs.
Case-1.2:
\[ A_0 = \frac{i\lambda B_0}{2\beta} \sqrt{3k^2 + 3p^2}, \]
\[ A_1 = \frac{i(2B_0 + \lambda B_0)}{2\beta} \sqrt{3k^2 + 3p^2}, \]
\[ A_2 = \frac{iB_0}{\beta} \sqrt{6k^2 + 6p^2}, r = 0.5k\left(k^2 + p^2\right)(\lambda^2 - 4\mu). \]

If we use the coefficients of Eq.(18) along with Eqs.(3) and (7) into Eq. (14), we obtain a new complex hyperbolic function solution to the modified Zakharov-Kuzentsov equation (ZK) with (2+1)-dimensions arising in plasma physics as following:
\[ u_z = \frac{i\sqrt{3k^2 + 3p^2}}{\sqrt{\beta}} \left(\tau + \lambda \sqrt{\tau} \text{tanh}[f]\right), \]
\[ \text{where } \tau = \lambda^2 - 4\mu, \text{ and } \]
\[ f = 0.5\sqrt{\tau} \left[E + kx + py + 0.5k\tau(k^2 + p^2)\right], \]
\[ \lambda^2 - 4\mu > 0 \text{ with the under the terms of Family-1}. \]

IV. Physical Expressions & Discussions and Remarks

In this subsection of paper, we introduce some basic properties of MEFM and the physical meaning of complex and hyperbolic function solutions obtained in this paper to the modified Zakharov-Kuzentsov equation (ZK) with (2+1)-dimensions arising in plasma physics.

Firstly, MEFM is more comprehensive according to the exp(\text{Omega}(xi)) -expansion method because MEFM is of one more parameter like \( M \). This gives so many coefficients, which leads to much more travelling wave solutions. As being in this paper, we have obtained so many analytical solutions to the modified Zakharov-Kuzentsov equation (ZK) with (2+1)-dimensions arising in plasma physics for only \( M = 1 \) and \( N = 2 \). If we take \( M = 2 \) and \( N = 3 \), we can write another following equations;
\[ U = \frac{A_1 + A_4 e^{3\Omega} + A_2 e^{2\Omega} + A_3 e^{\Omega}}{B_0 + B_1 e^{3\Omega} + B_2 e^{2\Omega}} = \frac{\Upsilon}{\Psi'}, \]
\[ U' = \frac{\Upsilon'\Psi - \Upsilon\Psi'}{\Psi'^2}, \]
\[ U'' = \cdots \]
\[ \vdots \]
where \( A_3 \neq 0, B_2 \neq 0 \). When we use Eq.(20) and Eq.(22) in Eq.(12), we can obtain new solutions to the model considered in this paper. Therefore, this procedure of Eq.(6) will contribute to obtain more travelling wave solutions and to better understanding of engineering and physical problems along with new physical predictions of models arising in plasma physics.

Secondly, hyperbolic functions are of many physical meanings as well. They arise in many problems of mathematics and mathematical physics. For instance, the hyperbolic sine arises in the gravitational potential of a cylinder \([10]\). The hyperbolic cosine function is the shape of a hanging cable. The hyperbolic tangent arises in the calculation of and rapidity of special relativity \([10]\).

If we consider our results, It is estimated that the hyperbolic solutions Eq.(17,19) are related to such physical meanings.

Finally, when it comes to the surfaces, Figures-1 and Figure-2 have been plotted by the Wolfram Mathematica 9 by considering the suitable values of parameters. These values of parameters are consistency with physical meaning of problem.

Conclusion

In this paper we have applied the application of MEFM to the modified Zakharov-Kuzentsov equation (ZK) with (2+1)-dimensions arising in plasma physics. We have obtained some new travelling wave solutions such as complex and hyperbolic functions solutions. We have observed that both solutions obtained in this paper verified to the Eq.(1) by using Wolfram Mathematica 9.
This method has provided many coefficients for Eq.(1). Both of them have been considered in this paper to obtain new analytical solutions. If other coefficients are considered, of course, one can obtain different prototype solutions to the Eq.(1).

Therefore, it can be said that this method is a powerful tool for obtaining solutions of the type as Eq.(1).

To the best of our knowledge, the application of MEFM to the Eq.(1) has not been submitted to literature in advance.

References