ENERGETIC APPROACH TO INVESTIGATION OF CHAOTIC BEHAVIOR OF LOW-DIMENSIONAL DYNAMIC SYSTEMS AND ITS ILLUSTRATION ON A TWO-DISC RIKITAKE DYNAMO

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Introduction. In the last few decades chaotic signals have widely been studied and applied to several fields [1]. In the case of zero-input continuous-time causal systems, the existence of chaos is impossible if the order \( n \) of that system is lower than 3. Further necessary condition for creation of the chaotic behavior is the existence of at least one strong non-linearity. It is well known that in a nonlinear third-order continuous-time system occurrence of a chaotic signal can be characterized by a single positive Lyapunov exponent [2], which demonstrates that the dynamics of the underlying chaotic attractor expands only in one direction. The chaotic phenomena with two and more positive Lyapunov exponents are often called hyperchaotic (this term was first introduced by Roessler [3]). Because one of the Lyapunov exponents is always zero and the sum of all Lyapunov exponents must be negative [4], it follows that the description of the hyperchaotic phenomena in a continuous-time system requires that the minimal order of the state space system representation, or equivalently, minimal number of independent energy storage elements must be at least four. Since then, a large number of various hyperchaos generating circuits have been proposed. Some of them have been designed as a chain of two or more non-linearly coupled second-order linear oscillators [5].

The paper presents the main ideas of a new signal-energy based approach to synthesis of controlled chaos and to a systematic structure oriented design method of chaotic and hyperchaotic systems, consisting of a finite number of nonlinearly coupled nonlinear oscillators.

One of the most essential characteristics of the chaotic behavior is its long-term non-predictability. Generally speaking, the chaotic behavior can be characterized as a non-periodic process in which irregular changes of instability intervals are followed by subintervals of dissipativity in such a way that in average the total system energy is conserved. Consequently, three attributes of reality are of crucial importance for the proposed approach: causality of non-linear system interactions, state energy conservation principle validity, and existence of non-linear instability region.

The proposed approach to chaotic phenomena is based on a generalization of the well known Tellegen theorem for electrical circuits [6], as a form of abstract energy conservation principle, combined with the famous second method of A. M. Lyapunov [7] as a fundamental tool to solve instability problems in context of chaos generation, chaos detection, chaos control, chaos synthesis and synchronization, from a unique physically plausible view point.

Mathematical model and its solution. Consider a continuous-time linear time-invariant strictly causal system representation \( R\{S\} \) of a system \( S \) given by

\[
R\{S\}: \quad \dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x^0, \quad y = Cx(t)
\]  

(1)
where $\mathbf{x}(t)$ represents the state vector with $\dim \mathbf{x}(t) = n$, $\mathbf{u}(t)$ represents the input signal vector, $\dim \mathbf{u}(t) = r$, and $\mathbf{y}(t)$ is the observed output signal vector with $\dim \mathbf{y}(t) = p$. A class of physically correct state equivalent finite-dimensional representations is generated by dissipation normal form introduced in [8].

The state space energy is defined as a measure of distance of the state $\mathbf{x}(t)$ from the zero equilibrium state $\mathbf{x}^* = 0$

$$E(x) = \frac{1}{2} \rho^2 \left[ \mathbf{x}(t), \mathbf{x}^* \right] = \frac{1}{2} \| \mathbf{x}(t) \|^2 = \frac{1}{2} \sum_{i=1}^{n} x_i^2(t)$$

and the power $P(t)$ of the output signal $\mathbf{y}(t)$ is defined by

$$P(t) = \| \mathbf{y}(t) \|^2.$$

Considering the zero input, the power balance relation can be expressed by

$$\frac{dE(x,t)}{dt} = -P(t) = -\| \mathbf{y}(t) \|^2$$

and hence the energy conservation principle for physically correct dissipative system representation can be expressed by

$$E(t_0) = \int_{t_0}^{\infty} \| \mathbf{y}(t) \|^2 \, dt.$$

The structure of the dissipation normal form can be interpreted as a chain of undamped linear or nonlinear oscillators with linear or nonlinear couplings with just one aggregated dissipation element. Such a structure has an important property – it is parametrically minimal. The dissipation of the whole system is concentrated in the dissipation parameter $\alpha_{11}$, expressing the aggregated dissipation in the direction of only one observed state component $x_{11}$. In some applications multidirectional dissipation can prove to be more useful. Let us consider a set of nonlinear oscillators with nonlinear couplings as follows.

$$A = \begin{bmatrix}
-\alpha_{11}(x) & \alpha_{12}(x) & 0 & 0 & 0 & 0 \\
-\alpha_{21}(x) & -\alpha_{22}(x) & \alpha_{23}(x) & 0 & 0 & 0 \\
0 & -\alpha_{31}(x) & -\alpha_{33}(x) & \ddots & \ddots & 0 \\
0 & 0 & \ddots & \ddots & \ddots & \alpha_{n-1}(x) & 0 \\
0 & 0 & 0 & -\alpha_{n-1}(x) & -\alpha_{n-1,n-1}(x) & \alpha_{n}(x) \\
0 & 0 & 0 & 0 & -\alpha_{n}(x) & -\alpha_{n,n}(x)
\end{bmatrix}$$

Again, the power interactions represented by components $\alpha_i$ are structurally neutral, but the dissipation of the system is now given by the trace of the matrix $A$.

Four important system classes can be studied:

a) system of linearly coupled nonlinear oscillators with multi-linear dissipation

b) system of nonlinearly coupled linear oscillators with multi-linear dissipation
c) system of nonlinearly coupled nonlinear oscillators with linear dissipation
d) system of nonlinearly coupled nonlinear oscillators with nonlinear dissipation

Consider a system of nonlinearly coupled oscillators with the following structure:

$$\begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2 \\
\ddot{x}_3 \\
\ddot{x}_4
\end{bmatrix} = \begin{bmatrix}
-\alpha_1 & \alpha_2 x_4 & 0 & 0 \\
-\alpha_2 x_4 & -k_1 & \alpha_3 & 0 \\
0 & -\alpha_3 & -k_2 & -\alpha_4 x_1 \\
0 & 0 & \alpha_4 x_1 & -\alpha_5
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
\beta_1 & 0 \\
0 & \beta_2 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}$$
One of many motivations for this structure comes from a standard hypothesis concerning the magneto-hydrodynamic explanation for chaotic changes of the Earth’s geomagnetic field. The couplings between the two oscillators are realized not only by the parameter $\alpha_3$ but by nonlinearity in the frequency parameters $\alpha_2 x_4^a$ and $\alpha_4 x_1^b$ as well. The system is supposed to be driven by two input signals $u_1$ and $u_2$ via parameters $\beta_1$ and $\beta_2$. The chaos generating parameters $k_1$ and $k_2$ have been fixed via state space energy correlation.

**Illustrative example: the two-disc Rikitake dynamo.** The two-disc Rikitake dynamo [9] represents a well-known and relatively simple system characterized by chaotic reversal. Its behavior has been studied for years [10, 11, 12] in association with the phenomena occurring in the geodynamo, in order to find possible physical analogies. And even when it was shown that such a simple system cannot simulate extremely complicated processes in the surface layers of the liquid Earth core, it is still recognized for its ability to demonstrate chaos and its relevant mathematical and physical aspects. Its basic arrangement is depicted in Fig. 1. The discs $D$ made from electrically conductive material (whose internal structure may, however, differ from case to case) are fixed on shafts $S$ driven by external mechanical torques $T_1(t)$ and $T_2(t)$. Their angular velocities are $\omega_1(t)$ and $\omega_2(t)$. The discs are connected to electric circuits $C$ that surround the other shaft. The currents $i_1(t)$ and $i_2(t)$ flowing in them produce in the loops $L$ magnetic fluxes $\Phi_{12}(t)$ and $\Phi_{21}(t)$ whose time changes induce voltages $u_1(t)$ and $u_2(t)$ between the shafts and rims of the discs (and represent, in this way, the voltage sources of the system).

![Fig. 1. The basic arrangement of the two-disc Rikitake dynamo](image)

The task represents an electromagnetically-mechanical coupled problem, whose electromagnetic part includes both field and circuit aspects. Provided that the system does not contain ferromagnetic parts, the circuit equations may be written in the following form.
\[ u_1(t) = R_1 i_1(t) + L_1 \frac{di_1(t)}{dt}, \]
\[ u_2(t) = R_2 i_2(t) + L_2 \frac{di_2(t)}{dt}, \]

while the equations of motion read
\[ J_1 \frac{d\omega_1(t)}{dt} + B_1 \omega_1(t) + T_{e1}(t) = T_1(t), \]
\[ J_2 \frac{d\omega_2(t)}{dt} + B_2 \omega_2(t) + T_{e2}(t) = T_2(t). \]

Here the symbols \( R_1 \) and \( R_2 \) denote the total resistances of circuits 1 and 2 (they consist of the resistances of the conductors, corresponding parts of the shafts and discs), \( L_1 \) and \( L_2 \) their inductances, \( J_1 \) and \( J_2 \) the moments of inertia of both shafts with discs, \( B_1 \) and \( B_2 \) the coefficients of the friction and \( T_{e1}(t) \) and \( T_{e2}(t) \) the electromagnetic torques.

Now the principal task is to determine the induced voltages \( u_1(t) \) and \( u_2(t) \), as well as the electromagnetic torques \( T_{e1}(t) \) and \( T_{e2}(t) \) acting on both discs. Let us start with the induced voltages. Suppose that the discs are sufficiently distant from each other, so that there is no other coupling between them. Now we can start from Fig. 1 showing a disc rotating at the angular velocity \( \omega \) in magnetic field of flux density \( B \). Now the electric field strength at any point \( Q \) of the disc of radius vector \( r \) is given as
\[ E = v \times B, \quad v = \omega \times r \]
where \( v \) is the actual velocity of point \( Q \). After some effort it can be shown that
\[ u_1(t) = -f_1(t) \cdot \Phi_{21}(t), \quad u_2(t) = -f_2(t) \cdot \Phi_{12}(t) \]
and the electromagnetic torque may be expressed as
\[ T_e(t) = -z_0 \cdot \frac{i(t)}{2\pi} \int_{\varphi=0}^{2\pi} \int_{r=R_c}^{R} B_c(t) \cdot r \cdot dr \cdot d\varphi = -z_0 \cdot \frac{i(t) \Phi(t)}{2\pi}. \]

In this way the dynamo equations may be modified as follows
\[ -M_{21} \frac{\omega_1(t)i_2(t)}{2\pi} = R_1 i_1(t) + L_1 \frac{di_1(t)}{dt}, \]
\[ -M_{12} \frac{\omega_2(t)i_1(t)}{2\pi} = R_2 i_2(t) + L_2 \frac{di_2(t)}{dt}, \]
\[ J_1 \frac{d\omega_1(t)}{dt} + B_1 \omega_1(t) - M_{21} \frac{i_1(t)i_2(t)}{2\pi} = T_1(t), \]
\[ J_2 \frac{d\omega_2(t)}{dt} + B_2 \omega_2(t) - M_{12} \frac{i_2(t)i_1(t)}{2\pi} = T_2(t). \]

The determination of the coefficients in system (13) is not quite easy. Problems may be with the damping factors and, particularly, with the self- and mutual inductances that must be calculated using the magnetic field approach.

On the other hand, chaotic behavior phenomenon is a structural problem that behavior can appear for broad class of parameters. The important problem in studying strong nonlinear systems is to determine a set of parameters for which system behaves chaotically. According to above definition the method based on using the abstract energy conservation principle was created. This method is demonstrated on the Rikitake system.
Rikitake system is an abstract system based on mathematical model of two-disc Rikitake dynamo. It is obvious that by a proper choice of state variables and after some manipulation it is possible to transform equations (13) into abstract form described by matrix equation (7).

Assume any choice of parameters \( (\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 0.01, \alpha_4 = 1, \alpha_5 = 1, \beta_1 = 1, \beta_2 = 1) \) and values of inputs \( (u_1 = 4, u_2 = 4) \). Now it is possible to ask whether (and for which values of parameters \( k_1, k_2 \)) the behavior of Rikitake system is chaotic. The parameters were chosen randomly to emphasize the structural properties of the chaotic behavior, even when it was possible to use parameters of any real Rikitake dynamo model. The proposed method uses autocorrelation of the abstract energy as a measure of chaoticity of the system. The simulation results for the given system are shown in Fig. 2. It is natural to expect that low the level of autocorrelation of the abstract energy implies chaotic behavior of system.

![Fig. 2. The relation between parameters \( k_1, k_2 \) and correlation of abstract energy](image)

It seems that the autocorrelation function of energy has minimum for \( k_1 \approx 0.077 \). In fact, the behavior of this system significantly differs in dependence on this parameter. The extreme sensitivity of system behavior on parameter \( k_1 \) value is demonstrated in Figs. 3 and 4. The behavior of the system is chaotic for \( k_1 = 0.077245 \) and stable for \( k_1 = 0.07724 \). It demonstrates one important feature of chaotic systems: extreme sensitivity to changes in parameters. Fig. 5 displays the evolution of all state variables of the chaotic system.

![Fig. 3. The projection of state trajectories of the Rikitake system:](image)

- a) \( k_1 = 0.077245 \)
- b) \( k_1 = 0.07724 \)
Conclusion. The paper deals with a general approach to studying chaotic systems. It was created a method based on a new measure of chaotic behavior – autocorrelation of abstract system energy. This method was demonstrated on the relatively simple example (4th order) – Rikitake system.

One of the advantages of the described approach is that the computational complexity does not depend directly on the order of the investigated system. Another interesting feature is that it is not necessary to provide any discretization of state space or state variable trajectories in contrast to other known methods (fractal dimensions, bifurcation diagrams etc.)

The goal of the future research should be a generalization of the signal energy approach to high order strongly non-linear systems analysis.

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SIMULATION OF INTERFERENCE PROCESSES OF VIBRATION CHARGE IN THE ELECTROMAGNETIC CONTOUR

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Здійснено інтегрування рівняння вимушeno коливання заряду в гармонійному електромагнітному осцилляторі. Було отримано точні вирази функцій обміну синусоїдного джерела і омічного опору для осциллятора енергії і визначено функцію кореляції з її спектром коливань.

Integrating the equation of forced vibration of the charge in the harmonic electromagnetic oscillator was carried out. The exact expressions for functions of oscillator energy exchange with sinusoidal source and Ohm’s resistors were obtained and function correlation with its oscillation spectrum was determined.

Introduction. Vibrations of the harmonic electromagnetic oscillator (HEMO) have been quite sufficiently studied, and the corresponding results have been generalised in a great number of monographs, including [1–4]. Nevertheless, the kinetics of the development of oscillator energy exchange with Ohm’s resistors and an external sinusoidal electromotion force (EMF) has not been thoroughly investigated, and this fact determined the objective of this work. Its topicality confirmed that relaxation processes of oscillator energy playing an important part not only in the physical [5, 6], but also at the study of more general issues [7, 8].

General relations. In this work, one-dimenssional vibrations of HEMO in the external sinusoidal force potential $U(t) = U_0 \left( \sin \Omega t \cos \Omega t \right)$ are considered. It is know, that vibration of charge are described by the differential equation

$$\frac{d^2 q(t)}{dt^2} + 2 \gamma \frac{dq(t)}{dt} + \omega_0^2 q(t) = \frac{U(t)}{L} \tag{1}$$

having solution