Abstract: The positive fractional and cone fractional continuous-time and discrete-time linear systems are addressed. Sufficient conditions for the reachability of positive and cone fractional continuous-time linear systems are given. Necessary and sufficient conditions for the positivity and asymptotic stability of the continuous-time linear systems are established. The realization problem for positive fractional continuous-time systems is formulated and solved.

Key words: cone, continuous-time, fractional, positive, realization problem.

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. The overview of a state-of-the-art situation in the field of positive systems is given in the monographs [8, 9]. The stability and robust stability of positive and fractional 1D linear systems has been investigated in many papers and books [1-9, 13, 23, 28]. Realization problem of a positive continuous-time and discrete-time linear system has been considered in [10, 12-15, 19, 20, 22]. Recently, the reachability, controllability and minimum energy control of positive linear discrete-time systems with time-delays have been considered in [9, 16-18, 21, 24].

The first definition of the fractional derivative was introduced by Liouville and Riemann at the end of the 19th century [50-52, 54, 55]. This idea was used by engineers for modeling different processes in the late 1960s. Mathematical fundamentals of fractional calculus are given in the monographs [23, 25-30]. The fractional order controllers were developed in [29]. Some other applications of fractional order systems can be found in [31, 32].

The main purpose of this paper is to give an overview of some recent results on positive and cone fractional continuous-time and discrete-time linear systems.

The paper is arranged as follows. In section 2 the positive fractional linear continuous-time systems are introduced. In section 3 the fractional cone systems are discussed. Sufficient conditions for the reachability are established in section 4. The realization problem for positive fractional continuous-time linear system is investigated in section 5. Positive fractional discrete-time linear systems are addressed in section 6. Sufficient conditions for the reachability of discrete-time linear systems are established in section 7. Concluding remarks are given in section 8.

The following notation will be used: \( \mathbb{R} \) - the set of real numbers, \( \mathbb{R}_{nn}^{m} \) - the set of \( n \times m \) real matrices, \( \mathbb{R}_{nn}^{m} \) - the set of \( n \times m \) matrices with nonnegative entries and \( \mathbb{R}_{n}^{m} = \mathbb{R}_{n+1}^{m} \), \( M_{n} \) - the set of \( n \times n \) Metzler matrices (real matrices with nonnegative off-diagonal entries), \( I_{n} \) - the \( n \times n \) identity matrix.

2. Positive fractional continuous-time linear systems

The following Caputo definition of the fractional derivative will be used [23, 25, 27, 29]

\[
\frac{d^{\alpha}}{dt^{\alpha}} f(t) = \frac{1}{\Gamma(k - \alpha)} \int_{0}^{t} f^{(k)}(\tau) (t - \tau)^{\alpha - k - 1} d\tau,
\]

where \( k - 1 < \alpha \leq k \in \mathbb{N} = \{1, 2, ..., \} \)

Consider the continuous-time fractional linear system described by the state equations

\[
\frac{d^{\alpha}}{dt^{\alpha}} x(t) = A x(t) + B u(t), \quad 0 < \alpha \leq 1,
\]

\[
y(t) = C x(t) + D u(t),
\]

where \( x(t) \in \mathbb{R}^{n}, \ u(t) \in \mathbb{R}^{m}, \ y(t) \in \mathbb{R}^{p} \) are the state, input and output vectors and \( A \in \mathbb{R}^{n \times n}, \ B \in \mathbb{R}^{n \times m}, \ C \in \mathbb{R}^{p \times n}, \ D \in \mathbb{R}^{p \times m} \).

Theorem 1. [23] The solution of equation (2.2a) is given by

\[
x(t) = \Phi_{\alpha}(t)x_{0} + \int_{0}^{t} \Phi(t - \tau) B u(\tau) d\tau, \quad x(0) = x_{0},
\]

where \( \Phi_{\alpha}(t) \) is the fractional solution of the homogeneous system.

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where
\[ \Phi_0(t) = E_a(A t^\alpha) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha + 1)}, \]  
\[ \Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k-1)\alpha}}{\Gamma((k+1)\alpha)}, \]  
and \( E_a(A t^\alpha) \) is the Mittag-Leffler matrix function, 
\[ \Gamma(x) = \int_0^\infty e^{-t} t^{-x} dt \]  
the gamma function.

**Definition 1.** [23] The system (2) is called the internally positive fractional system if and only if the matrix \( A \) is a Metzler matrix and every state \( x \in \mathbb{R}^n_+ \) is reachable in time \( t \in [0, t_f] \) if for every initial condition \( x_0 \in \mathbb{R}^n_+ \) and inputs \( u(t) \in \mathbb{R}^m_+ \) for \( t \geq 0 \) there exists such a time \( t_f \) that the state \( x(t) \) steers the state of the system (2) from zero initial state \( x_0 = 0 \) to the desired state \( x_f \) in time \( t_f \).

**Theorem 2.** [23] The continuous-time fractional system (2) is internally positive if and only if the matrix \( A \) is a Metzler matrix and \( A \in M_n_+ \), \( B \in \mathbb{R}^{m \times n}_+ \), \( C \in \mathbb{R}^{p \times n}_+ \), \( D \in \mathbb{R}^{p \times m}_+ \).

**3. Cone fractional systems**

Following [10, 23] the definitions are recalled.

**Definition 2.** Let \( P = \begin{bmatrix} p_1 & \cdots & p_n \end{bmatrix} \in \mathbb{R}^{n \times n}_+ \) be nonsingular and \( p_k \) be the \( k \)-th \((k = 1, 2, \ldots, n)\) its row.

The set
\[ \Phi := \left\{ x \in \mathbb{R}^n : \sum_{k=1}^n p_k x \geq 0 \right\} \]  
is called the linear cone generated by the matrix \( P \).

In a similar way we may define the linear cone \( \mathcal{Q} := \left\{ u \in \mathbb{R}^m : \sum_{k=1}^m q_k u \geq 0 \right\} \) generated by the nonsingular matrix \( Q = \begin{bmatrix} q_1 & \cdots & q_m \end{bmatrix} \in \mathbb{R}^{m \times m}_+ \).

The inputs \( u \), and the linear cone \( \Phi := \left\{ y \in \mathbb{R}^p : \sum_{k=1}^p v_k y \geq 0 \right\} \) generated by the nonsingular matrix \( V = \begin{bmatrix} v_1 & \cdots & v_p \end{bmatrix} \in \mathbb{R}^{p \times p}_+ \) for the outputs \( y \).

**Definition 3.** The fractional system (2) is called the cone fractional system if \( x(t) \in \Phi \) and \( y(t) \in \mathcal{V} \), \( t \geq 0 \) for every \( x_0 \in \Phi \) and \( u(t) \in \mathcal{Q} \), \( t \geq 0 \).

The \((\Phi, \mathcal{Q}, \mathcal{V})\) cone fractional system (2) will be shortly called the cone fractional system. Note that if \( \Phi = \mathbb{R}^n_+ \), \( \mathcal{Q} = \mathbb{R}^m_+ \), \( \mathcal{V} = \mathbb{R}^p_+ \), then the \((\mathbb{R}^n_+, \mathbb{R}^m_+, \mathbb{R}^p_+)\) cone system is equivalent to the classical positive system [18, 26].

**Theorem 3.** The fractional system (2) is \((\Phi, \mathcal{Q}, \mathcal{V})\) a cone fractional system if and only if
\[ \tilde{A} = P A P^{-1} \in \mathbb{R}^{n \times n}_+, \tilde{B} = P B Q^{-1} \in \mathbb{R}^{m \times n}_+, \]  
\[ \tilde{C} = V C P^{-1} \in \mathbb{R}^{p \times m}_+, \tilde{D} = V D Q^{-1} \in \mathbb{R}^{p \times m}_+ \]
Proof is given in [23, 17].

**3. Reachability of positive fractional systems**

**Definition 4.** The state \( x_f \in \mathbb{R}^n_+ \) of the fractional system (2) is called reachable in time \( t_f \) if there exists an input \( u(t) \in \mathbb{R}^m_+ \), \( t \in [0, t_f] \) which steers the state of system (2) from zero initial state \( x_0 = 0 \) to the state \( x_f \). If every state \( x_f \in \mathbb{R}^n_+ \) is reachable in time \( t_f \), the system is called reachable in time \( t_f \). If for every state \( x_f \in \mathbb{R}^n_+ \) there exists such a time \( t_f \) that the state is reachable in time \( t_f \), the system (2) is called reachable.

A real square matrix is called monomial if and only if each its row and column contains only one positive entry and the remaining entries are zero.

**Theorem 4.** The continuous-time fractional system (2) is reachable in time \( t_f \) if the matrix
\[ R(t_f) = \int_0^{t_f} \Phi(\tau) B T^\tau D \Phi^T(\tau) d\tau \]  
is a monomial matrix.

The input which steers the state of the system (2) from \( x_0 = 0 \) to \( x_f \) is given by the formula
\[ u(t) = B T^\tau (t_f - t) R^{-1}(t_f) x_f \]
where \( T \) denotes the transposition. A proof is given in [21].

**Definition 5.** A state \( x_f \in \Phi \) of the cone fractional system (2) is called reachable in time \( t_f \) if there exists an input \( u(t) \in \mathcal{Q} \), \( t \in [0, t_f] \) which steers the state of the system from zero initial state \( x_0 = 0 \) to the desired state \( x_f \). If every state \( x_f \in \Phi \) is reachable in time \( t_f \), then the cone fractional system is called reachable in time \( t_f \). If for every state \( x_f \in \Phi \) there exists a time \( t_f \), then the cone fractional system is called reachable.
Theorem 5. The cone fractional system (2) is reachable in time $t_f$ if and only if the matrix
$$ R(t_f) = \int_0^{t_f} \Phi(r)BQ^{-1}Q^{-T}B^T \Phi^T(r) d\tau P^T(Q^{-1})^T $$ (13)
is a monomial matrix. A proof is given in [21].

From Theorem 5 we have the following corollary.

Corollary 1. If $Q = I_m$, then $R(t_f) = PR(t_f)P^T$, and the cone fractional system (2) is reachable in time $t_f$ if the positive fractional system is reachable and $P$ is a monomial matrix.

Example 1. Consider the cone fractional system (2) with
$$ P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \end{bmatrix}. $$ (14)

The $\mathcal{P}$-cone generated by the matrix $P$ is shown in Fig. 1.

It is easy to show that
$$ \Phi(t)B = \begin{bmatrix} 0 & \Phi_1(t) \\ \Phi_2(t) & 0 \end{bmatrix} $$ (15)
and
$$ R(t_f) = \int_0^{t_f} \Phi(r)BB^T \Phi^T(r) d\tau = \int_0^{t_f} \begin{bmatrix} \Phi_1^2(r) & 0 \\ 0 & \Phi_2^2(r) \end{bmatrix} d\tau $$ (16)
where
$$ \Phi_1(t) = \sum_{k=0}^{\infty} \frac{t^{k+1}\alpha^{k+1}}{\Gamma(k+1)\alpha} \Phi_2(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}, \quad 0 < \alpha < 1 $$ (17)
The matrix (16) is monomial and according to Theorem 4 the positive fractional system is reachable in time $t_f$. In the case $Q = I_2$ the matrix
$$ \tilde{R}(t_f) = PR(t_f)P^T = \int_0^{t_f} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \Phi_1^2(r) & 0 \\ 0 & \Phi_2^2(r) \end{bmatrix} d\tau = \int_0^{t_f} \begin{bmatrix} \Phi_1^2(r) + \Phi_2^2(r) & \Phi_1^2(r) - \Phi_2^2(r) \\ \Phi_2^2(r) - \Phi_1^2(r) & \Phi_1^2(r) + \Phi_2^2(r) \end{bmatrix} d\tau $$ (18)
is not monomial, since $\Phi_1^2(r) \neq \Phi_2^2(r)$.

Therefore, the sufficient condition for the reachability in time $t_f$ of Theorem 5 is not satisfied.

From this example and comparison of (11) and (13) it follows that the sufficient condition for the reachability of the cone fractional systems is much stronger than for the positive fractional systems.

A state $x_0 \in \mathcal{P}$ of the cone fractional system (2) is called controllable to zero in time $t_f$ if there exist an input $u(t) \in \mathcal{Q}$, $t \in [0, t_f]$ which steers the state of the system from $x_0$ to the zero state $x_f = 0$. Following [26] it is possible to extend the considerations to the controllability to zero of the cone fractional linear system.

4. Realization problem for positive fractional systems
Consider the continuous-time fractional linear system described by the state equations
$$ \frac{d^\alpha}{dt^\alpha} x(t) = Ax(t) + Bu(t), \quad 0 < \alpha \leq 1 $$ (19a)
$$ y(t) = Cx(t) + Du(t) $$ (19b)
where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the state, input and output vectors and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$.

Applying the Laplace transform to (19), it is easy to show that the transfer matrix of the system is given by the formula
$$ T(s) = C[I - s^\alpha A]^{-1} B + D. $$ (20)
The transfer matrix is called proper if and only if
$$ \lim_{s \to \infty} T(s) = K \in \mathbb{R}^{p \times m} $$ (21)
and it is called strictly proper if and only if $K = 0$.

From (20) we have
$$ \lim_{s \to \infty} T(s) = D $$ (22)
since
$$ \lim_{s \to \infty} [I - s^\alpha A]^{-1} = 0. $$ (23)

Definition 6. Matrices $A$, $B$, $C$, $D$ are called a positive fractional realization of a given transfer matrix.
\( T(s) \) if they satisfy the equality (20). A realization is called minimal if the dimension of \( A \) is minimal among all realizations of \( T(s) \).

The positive fractional realization problem can be stated as follows. Being given a proper transfer matrix \( T(s) \), find its positive realization.

First the realization problem will be solved for single-input single-output (SISO) linear fractional systems with all realizations of \( T(s) \). called minimal if the dimension of \( A \) is minimal.

Theorem 6. There exist positive fractional minimal realizations of the form

\[
T(s) = \frac{b_k}{s^a} + \sum_{i=1}^{n-1} \left( \frac{a_i}{s^{a+i}} \right) + \frac{a_{n-1}}{s^{a+n-1}} - a_0 s^a - a_0
\]

Using (22), we obtain

\[
D = \lim_{s \to \infty} T(s) = b_n,
\]

and the strictly proper transfer function has the form

\[
T_{sp}(s) = T(s) - D = \frac{\bar{b}_n}{s^a} + \sum_{i=1}^{n-1} \left( \frac{a_i}{s^{a+i}} \right) + \frac{a_{n-1}}{s^{a+n-1}} - a_0 s^a - a_0
\]

where

\[
\bar{b}_k = b_k + a_kb_n, \quad k = 0,1,\ldots,n-1.
\]

From (27) it follows that if \( a_k \geq 0 \) and \( b_k \geq 0 \) for \( k = 0,1,\ldots,n \), then also \( \bar{b}_k \geq 0 \) for \( k = 0,1,\ldots,n-1 \).

The matrices (28) are minimal realizations if and only if the transfer function (24) is irreducible.

Proof is given in [37].

The matrices (28) are minimal realizations if and only if the transfer function (24) is irreducible.

If the conditions of Theorem 6 are satisfied then the positive minimal realizations (28) of the transfer function (24) can be computed by use of the following procedure.

Procedure 1.

Step 1. Knowing \( T(s) \) and using (25), find \( D \) and the strictly proper transfer function (26).

Step 2. Using (28), find the desired realizations.

Example 1. Find the positive minimal fractional realizations (28) of the irreducible transfer function

\[
T(s) = \frac{2(s^a)^2 + 5s^a + 1}{(s^a)^2 + 2s^a - 3}.
\]

Using Procedure 1 and (29) we obtain the following:

Step 1. From (25) and (29) we have

\[
D = \lim_{s \to \infty} \frac{2(s^a)^2 + 5s^a + 1}{(s^a)^2 + 2s^a - 3} = 2
\]

and

\[
T_{sp}(s) = T(s) - D = \frac{s^a + 7}{(s^a)^2 + 2s^a - 3}.
\]

Step 2. Taking into account that in this case \( \bar{b}_0 = 7, \bar{b}_1 = 1 \) and using (28), we obtain the desired positive minimal fractional realizations

\[
A = \begin{bmatrix} a_{n-1} & 1 & 0 & 0 & 0 \\ a_{n-2} & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1 & 0 & 0 & 1 & 0 \\ a_0 & 0 & 0 & 0 & 1 \end{bmatrix},
B = \begin{bmatrix} \bar{b}_n \\ \bar{b}_{n-1} \\ \vdots \\ \bar{b}_1 \\ \bar{b}_0 \end{bmatrix},
C = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, D = b_n,
\]

(28d)

(28e)
Let’s consider a multi-input multi-output (MIMO) positive fractional system (19) with a proper transfer matrix $T(s)$.

Using the formula

$$D = \lim_{s \to \infty} T(s)$$  \hspace{1cm} (33)

we can find the matrix $D$ and the strictly proper transfer matrix which can be written in the form

$$T_{sp}(s) = T(s) - D = \begin{bmatrix} N_{11}(s) & N_{1m}(s) & \cdots & N_{1d_k}(s) \\ D_{1}(s) & \cdots & \cdots & D_{m}(s) \end{bmatrix} = N(s)D^{-1}(s), \hspace{1cm} (34)$$

Where

$$N(s) = \begin{bmatrix} N_{11}(s) & \cdots & N_{1m}(s) \\ \vdots & \ddots & \vdots \\ N_{p1}(s) & \cdots & N_{pm}(s) \end{bmatrix},$$

$$D = \text{diag}[D_1(s), \ldots, D_m(s)]$$  \hspace{1cm} (35)

$$N_{ik}(s) = c_{ik}^{-1}(s^\alpha)^{d_k-1} + \cdots + c_{ik}^{1}\alpha + c_{ik}^{0}, \quad i = 1, \ldots, p; \quad k = 1, \ldots, m$$

$$D_k(s) = (s^\alpha)^{d_k} - a_{k_d}d_k - \cdots - a_{k1}\alpha - a_{k0}$$  \hspace{1cm} (36)

**Theorem 7.** There exists the positive fractional realization

$$A = \text{block diag} [A_1, \ldots, A_m] \in \mathbb{R}^{n \times n},$$

$$A_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_{k0} & -a_{k1} & -a_{k2} & \cdots & -a_{kd_k} \end{bmatrix} \in \mathbb{R}^{d_k \times d_k},$$

$$B = \text{block diag} [B_1, \ldots, B_m] \in \mathbb{R}^{n \times m},$$

$$B_k = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ c_{11}^{0} & c_{11}^{1} & \cdots & c_{1m}^{0} & \cdots & c_{1m}^{d_m-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ c_{p1}^{0} & c_{p1}^{1} & \cdots & c_{p1}^{d_m-1} & \cdots & c_{pm}^{0} & \cdots & c_{pm}^{d_m-1} \end{bmatrix} \in \mathbb{R}^{p \times n}.$$  \hspace{1cm} (37)

of the transfer matrix $T(s)$ if the following conditions are satisfied:

i) $T(\infty) \in \mathbb{R}^{p \times m}$

ii) $a_{kl} \geq 0$ for $k = 1, \ldots, m, \quad l = 0, 1, \ldots, d_k - 1$ and $a_{kd_k} - 1$ can be arbitrary

iii) $c_{ik}^{j} \geq 0$ for $i = 1, \ldots, p; \quad j = 0, 1, \ldots, d_k - 1; \quad k = 1, \ldots, m$

A proof is given in [22].

If the conditions of Theorem 7 are satisfied, then the positive fractional realization of the transfer matrix $T(s)$ can be computed by use of the following procedure.

**Procedure 2.**

Step 1. Knowing the proper transfer matrix $T(s)$ and using (33), compute the matrix $D$ and the strictly proper matrix $T_{sp}(s)$.

Step 2. Find the minimal degrees $d_1, \ldots, d_m$ of the denominators $D_1(s), \ldots, D_m(s)$ and write the matrix $T_{sp}(s)$ in the form (34).

Step 3. Using the equality

$$D(s) = \text{diag}[(s^\alpha)^{d_1}, \ldots, (s^\alpha)^{d_m}] - \text{diag}[a_{10}, a_{21}, \ldots, a_{kd_k-1}]$$

find $A_k = [a_{k0}, a_{k1}, \ldots, a_{kd_k-1}]$ for $k = 1, \ldots, m$ and the matrix $A$.

Step 4. Knowing the matrix $N(s)$ and using

$$N(s) = C \in \mathbb{R}^{n \times p},$$

$$C = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & s^\alpha & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s^\alpha \end{bmatrix} \in \mathbb{R}^{p \times n},$$

find the matrix $C$.  \hspace{1cm} (39)

**Example 2.** Find the positive fractional realization (37) of the transfer matrix

$$T(s) = \begin{bmatrix} 2s^\alpha + 1 \\ s^\alpha + 1 \end{bmatrix}$$

$$\begin{bmatrix} 2s^\alpha + 1 \\ s^\alpha + 1 \end{bmatrix}$$

Using the Procedure 2, we obtain the following.

Step 1. From (33), (34) and (40) we have

$$D = \lim_{s \to \infty} T(s) = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \hspace{1cm} (41)$$

Step 2. Find the minimal degrees $d_1, d_2$ of the denominators $D_1(s), D_2(s)$ and write the matrix $T_{sp}(s)$ in the form (34).

$$T_{sp}(s) = T(s) - D = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

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5. Positive fractional discrete-time systems

In this paper the following definition of the fractional discrete derivative will be used

$$\Delta^\alpha x_k = \sum_{j=0}^{\alpha-1} (-1)^j \binom{\alpha}{j} x_{k-j}, \quad 0 < \alpha < 1$$  (50)

where $\alpha \in \mathbb{R}$ is the order of the fractional difference, and

$$\left( \begin{array}{c} a_1 \\ a_2 \end{array} \right) = \left( \begin{array}{c} 1 \\ \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} \end{array} \right)$$  for $j = 1, 2, \ldots$

Consider the fractional discrete linear system described by the state-space equations

$$\Delta^\alpha x_{k+1} = Ax_k + Bu_k, \quad k \in \mathbb{Z}_+$$  (52a)

$$y_k = Cx_k + Du_k$$  (52b)

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^p$ are the state, input and output vectors and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$.

Using the definition (50) we may write the equations (52) in the form

$$x_{k+1} + \sum_{j=0}^{\alpha-1} (-1)^j \binom{\alpha}{j} x_{k-j} = A x_k + B u_k, \quad k \in \mathbb{Z}_+$$  (53a)

$$y_k = C x_k + D u_k$$  (53b)

Definition 7. The system (53) is called the (internally) positive fractional system if and only if $x_0 \in \mathbb{R}^n_+$ and $y_k \in \mathbb{R}^p_+$, $k \in \mathbb{Z}_+$ for any initial conditions $x_0 \in \mathbb{R}^n_+$ and all input sequences $u_k \in \mathbb{R}^m_+$, $k \in \mathbb{Z}_+$.

Theorem 8. The solution of equation (53a) is given by

$$x_k = \Phi_k x_0 + \sum_{i=0}^{\alpha-1} \Phi_{k-i-1} B u_i$$  (54)

where $\Phi_k$ is determined by the equation

$$\Phi_{k+1} = (A + I \alpha) \Phi_k + \sum_{i=2}^{\alpha-1} (-1)^{i-1} \binom{\alpha}{i} \Phi_{k-i+1}$$  (55)

with $\Phi_0 = I_n$.

The proof is given in [16, 23].

Lemma 1. [16] If

$$0 < \alpha \leq 1$$  (56)

then

$$(-1)^{i-1} \binom{\alpha}{i} > 0 \quad \text{for } i = 1, 2, \ldots$$  (57)
6. Reachability of positive fractional linear systems

Consider the positive fractional linear system (53).

Definition 8. A state \( x_f \in \Re^n_+ \) of the positive fractional system (53) is called reachable in \( q \) steps if there exist a sequence \( u_k \in \Re^n_+ \), \( k = 0,1,\ldots,q-1 \) which steers the state of the system from zero \((x_0 = 0)\) to the final state \( x_f \), i.e. \( x_f = x_q \).

Let \( e_i, i = 1,\ldots,n \) be the \( i \)-th column of the identity matrix \( I_n \). A column \( ae_i \) for \( a > 0 \) is called the monomial column.

Theorem 10. The positive fractional system (53) is reachable in \( q \) steps if and only if the reachability matrix

\[ R_q = [B, \Phi,B,\ldots,B] \]  

contains \( n \) linearly independent monomial columns.

Proof. Using (54) for \( k = q \) and \( x_0 = 0 \) we obtain

\[ x_f = x_q = \sum_{i=0}^{q-1} \Phi q-iBu_i = R_q \begin{bmatrix} u_{q-1} \\ u_{q-2} \\ \vdots \\ u_0 \end{bmatrix} \]  

From Definition 8 and (60) it follows that for every \( x_f \in \Re^n_+ \) there exist an input sequence \( u_k \in \Re^n_+ \), \( i = 0,1,\ldots,q-1 \) if and only if the matrix (59) contains \( n \) linearly independent monomial columns. □

From (5.6) it follows that for positive fractional systems the coefficients \( a_i, i = 0,1,\ldots,k-1 \) in the equality

\[ \Phi_k = (A + I_\alpha)^k + a_k, (A + I_\alpha)^{k-1} + \ldots + a_1(A + I_\alpha) + a_0I_n \]  

are nonnegative.

Theorem 11. The positive fractional system (53) is reachable only if the matrix

\[ [A + I_\alpha, B] \]  

is not reachable in spite of that in this case the matrix

\[ [A + I_\alpha, B] = \begin{bmatrix} 1+\alpha & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]  

contains two linearly independent monomial columns.

The following example shows that for positive fractional systems the matrix (59) in Theorem 10 can not be substituted by the matrix

\[ R_q = [B, (A + I_\alpha)B,\ldots,(A + I_n\alpha)^{-1}B] \]

Example 4. Consider the positive fractional system (53) with the matrices

\[ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\alpha & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]  

In this case

\[ A + I_\alpha = \begin{bmatrix} \alpha & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \in \Re^{3 \times 3}_+ \]  

and the matrix (65) has the form

\[ R_3 = [B, (A + I_\alpha)B, (A + I_n\alpha)^{-1}B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \]

and it contains three linearly independent monomial columns. Using (55) for \( k = 0,1 \) for (66) we obtain

\[ \Phi_1 = (A + I_\alpha) = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

\[ \Phi_2 = (A + I_\alpha)\Phi_1 = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

and the matrix (59) has the form

\[ R_3 = [B, \Phi_1B, \Phi_2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix} \]  

which contains only two linearly independent monomial columns.

Definition 9. Let the \( j \)-th column \( b_j \) \( (j = 1,\ldots,m) \) of the matrix \( B \) be monomial. The column \( \Phi b_j \) \( (j = 1,\ldots,n) \) is positive if and only if the number of linearly independent monomial columns of (59) can be not greater than of the matrix (62). □

The following example shows that the condition of Theorem 11 is necessary but not sufficient.

Example 3. It is easy to show that the positive fractional system (53) with the matrices

\[ A = \begin{bmatrix} 1 & 0 \\ 0 & -\alpha \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{for} \quad 0 < \alpha < 1 \]

contains at least \( n \) linearly independent monomial columns.

Proof. From the form of the matrix (59) and the equality (61) it follows that the number of linearly independent monomial columns of (59) can not be greater than of the matrix (62). □
of the matrix $\Phi_1$ is called monomial column corresponding to the $j$-th column of $B$ if and only if it is monomial and linearly independent of the monomial column $b_j$.

In the new test for checking the reachability of the positive fractional systems a crucial role will play the following procedure [11].

Procedure 3. (finding linearly independent monomial columns).

Using Definition 9 find all monomial linearly independent columns (starting from the first column of $B$)

$$\Phi_{ij} = \Phi_i b_j \quad \text{for} \quad j = 1, \ldots, m; \quad k = 2, \ldots, q - 1$$ (71)

of the matrix (59). Stop the procedure if the last column is not monomial or/and linearly dependent from the previous monomial columns.

**Theorem 12.** The positive fractional system (53) is reachable if and only if using Procedure 3 to the matrix (59) it is possible to find its $n$ monomial linearly independent columns.

**Proof.** By Theorem 10 the positive fractional system (53) is reachable in $q$ steps if and only if the reachability matrix (59) contains $n$ monomial linearly independent columns. Thus, the system is reachable if and only if using the procedure it is possible to find $n$ monomial linearly independent columns of the matrix (59).

**Example 5.** Consider the positive fractional system (53) with the matrices

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & a & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ (72)

for $a > 0$.

It is easy to shown that for $a \neq 0$, rank$[B, AB, A^2 B] = 3$ and the standard (nonpositive) system is reachable in $q = 3$ steps. Now it will be shown that the positive fractional system (53) with (72) for $a > 0$ is unreachable. Using the Procedure 3 for the matrix

$$R_3 = [B, \Phi_1 B, \Phi_2 B] =$$

$$= B (A + I_n \alpha) B, (A + I_n \alpha)^2 B - \left(\frac{\alpha}{2}\right) B$$ (80)

we obtain only one monomial column $B$, since

$$B (A + I_n \alpha) B = \begin{bmatrix} \alpha \\ 1 \\ 0 \end{bmatrix}, \quad (A + I_n \alpha)^2 B - \left(\frac{\alpha}{2}\right) B = \begin{bmatrix} \alpha(\alpha + 1) \\ 2 \alpha \\ \alpha \end{bmatrix}$$ (81)

Thus, the positive fractional system is unreachable.
In the case b) using (59) and (85) we obtain the matrix

$$[B, \Phi_1 B, \Phi_2 B, \ldots] = \begin{bmatrix}
0 & 0 & a_{12} & \cdots \\
0 & 1 & a_{22} & \cdots \\
0 & 0 & a_{23} & \cdots \\
1 & 0 & a_{24} + \frac{a(a-1)}{2} & \cdots
\end{bmatrix} \quad (87)$$

with only two monomially independent columns. By Theorem 10 in this case the positive fractional system is unreachable.

It is well-known that the observability is a dual notion to the reachability. All results presented in this section for the reachability of positive fractional systems can be applied for checking the observability of the positive fractional systems.

7. Concluding remarks

The positive fractional linear continuous-time systems have been addressed. The cone fractional linear systems have been introduced. Sufficient conditions for the reachability of positive fractional and cone fractional linear systems have been established. The realization problem for positive fractional linear continuous-time systems has been formulated and solved. The positive fractional discrete-time linear systems are also considered.

Extensions of these considerations for the following classes of systems are open problems

1) 1D and 2D varying positive linear systems
2) 2D hybrid systems without and with delays
3) 2D Lyapunov systems
4) 1D and 2D positive fractional switching systems.

References


ДОДАТНІ ДРОБОВІ ТА КОНІЧНІ ДРОБОВІ ЛІНІЙНІ СИСТЕМИ

Тадеуш Качорек

У статті розглянуто додатні дробові та конічні дробові неперервні та дискретні лінійні системи. Наведено достатні умови для досягнення таких систем. Встановлено необхідні та достатні умови для додатністі та асимптотичної стабільності неперервних у часі лінійних систем.

Сформульовано та розв’язано проблему реалізації додатних дробових неперервних у часі систем.

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