MATHEMATICAL MODELING AND COMPUTING, Vol. 1, No. 1, pp. 97–108 (2014)

To modeling the auxetic materials: some fundamental aspects

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(Received 31 May 2014)

The auxetic materials are considered from the point of view of correspondence to the classical theory of elasticity. It is shown that some classical postulates relative to the elastic constants should be refined. Three cases of description of auxetic materials — by the model of linear elastic isotropic body, by the model of linear elastic transversally isotropic body, by the nonlinear elastic isotropic body (Murnaghan potential) — are analyzed shortly. The initial assumption on positivity of internal energy of deformation is saved and then the uniform stress states (unilateral tension, omnilateral compression, pure shear) are used to analyze the elastic constants. This allows to describe the new mechanical effects: expansion of the standard sample-rod-prism under unilateral tension and expansion of the standard sample-cube under hydrostatic compression as well as an existence of the arbitrary negative values of Poisson ratios, what is accompanied by the negative values of the Lame λ, Young E and compression k moduli, for the linear isotropic case and some elastic constants in the linear transversely isotropic case. The case of nonlinear description shows that the auxetic materials should be defined by the primary physical effect — observation in the standard for mechanics of materials experiment of longitudinal tension of a prism that the transverse deformation of prism is positive (a material as if swells) in contrast to the classical materials, where it is negative.

Keywords: *auxetic material, isotropic material, anisotropic material, negative values of elastic moduli, restriction on elastic moduli*

2000 MSC: 74B99

UDC: 539.3

1. Introduction

The mechanics of materials is still developing area of mechanics. It is well-known that it started at beginning of the nineteenth century with analysis of the simplest model of the deformation of materials — the model of linearly elastic isotropic deformation. Since then a lot of different models have been offered: from the basic today models of thermoelastic, viscoelastic, plastic, electroelastic, magnetoelastic deformation and so on to their combinations like the thermoelastoviscopiezomagnetoplastic model. Also, another specific models were proposed, sometimes very exotic (see, for example, the Mindlin microstructural model or the Drumheller-Sutherland lattice model in mechanics of composite materials) [1]. The nonlinear models formed later a big fragment of models in mechanics of materials, too [2–5]. With time, the primary model of isotropic material was supplemented with more complicate models of transversally isotropic and orthotropic materials, which afterwards were extended on the studies first of all in theory of crystals different cases of anisotropy (see, for example, Lekhnitsky [6] or Dielesaint-Royer [7] books). Last decades, the models in structural mechanics of materials are actively developed as applied to materials of macro-, meso-, micro- and nanolevels [8–10].

This general situation with presence of plenty of models allows to Truesdell [11] to formulate the sentence that, in fact, mechanics is an infinite class of models to represent certain aspects of nature.

But the primary model of elastic deformation was always the most studied and all the principal moments in constructing the models were elaborated on this model. Therefore the classical model of elastic body is considered as the standard one and working up fundamental facts for this model are not calling in question.

A few decades ago, this situation starts to change owing to, first of all, the progress in producing the new materials. First, the new facts in mechanics of composite materials were appeared, which showed that the notion of physical constants of composite materials is not inviolable: two models for one and the same material can be absolutely different and contain absolutely different sets of physical constants. Second, the new kinds of materials were produced, which extend the classical concepts and need some revision of the classical models. One of examples of this last observation is discussed further.

2. On auxetic materials as new kind of elastic materials

The auxetic materials are commonly thought as materials with negative Poisson ratio (mostly denoted as "nu" ν and sometimes as "sigma" σ) and introduced into scientific literature for last ten years. Because they are studied mainly by scholars working in molecular physics and material science, then such definition is there quite sufficient. But for mechanics, this definition is to fuzzy. First of all, the Poisson ratio is an elastic constant, introduced in the classical linear isotropic theory of elasticity to express the Poisson law. This law was formulated as follows: when in a long linearly elastic rod or beam the uniform stress-state is formed by tension of the rod at the ends (the rod is stretched and longitudinal strain is positive), then the transverse strain becomes negative (the rod cross-section becomes smaller and the rod constricts). When the Poisson ratio is negative, this means that the transverse strain becomes positive (the rod cross-section increases and the rod swells). This explains the name for auxetic materials — the Greek word $\alpha\upsilon\xi\eta\tau\iota\kappa o\zeta$ means "that which tends to increase". The auxetic materials are sometimes named "auxetics".

Thus, the existence of auxetic materials contradicts the Poisson law and therefore, contradicts the established rule of mechanics of materials. This contradiction can be considered as conditional, because many authors of classical monographs and textbooks on the theory of elasticity [12–18] shows one and the same restrictions on values of Poisson ratio

$$-1 < \nu < (1/2) \tag{1}$$

This fact will be discussed further and now the only statement should be shown that practically all researchers of auxetics (working mainly in molecular physics and material sciences) trust firmly in correctness of the shown before restrictions (the most cited book is the Landau and Lifshits one [16]). It is curiously that at the same time, the values essentially less of -1 do not to provoke objections by these researchers (see, for example, reviews of modern studies of auxetic materials [19] (2012) and [20] (2013)).

In should be noted just here that hypothetic possibility of auxetic materials was discussed essentially earlier in some books on the theory of elasticity (for example, in the Love's [13, page 104] and Lurie's books [2, page 117]. But they were also based on validity of restriction (1), where the values less of -1 and more than $(1/2)$ are forbidden.

Return now to the real auxetic materials. It is assumed commonly that the first mentions of materials with negative Poisson ratio can be found in publications of Gibson with co-authors of 1982 [21,22]. The first description of the real foam-type materials with negative Poisson ratio was reported practically simultaneously in 1987 by Lakes [23] and Wojciechowski [24]. The term "auxetic material" was introduced by Evans in 1991 [25]. Let us stop on the fact that the first virtually constructed and really observed auxetics were the foam-type internal structure. This feature just as the property to increase in transverse direction under longitudinal loading can be meant as the result of special internal structure of material, which in the most cases is described by an anisotropic continuum. It can be easily seen that different materials shown in the modern publications devoted to auxetics have the internal structure which can be modeled by isotropic medium with great reserve. But in some cases, the continuum can be considered as conditionally isotropic and then a definition of the auxetic as the material with negative Poisson ratio becomes conditionally correct.

And what will be not correct? Quick answer: the definition of auxetics as not isotropic materials with negative Poisson ratio will be incorrect. Because, in this case, each type of symmetry of material will be characterized by its number of different Poisson ratios. For example, the transversally isotropic elastic continua are described by three Poisson ratios, the orthotropic ones — by six Poisson ratios and so on. Indeed, they are not all independent and some of them can be negative and others positive. Such facts are described in recent publications (for example, some crystals are auxetic in one direction and not auxetic in other ones).

The most important results in creating the new auxetics is that the Poisson ratio has different negative values from -0.7 for the foam in the paper [4] what can be adjusted to the classical restrictions (1), to -12.0 for the polymer polytetrafluoroethylen in the paper [1], what goes far out from the classical rule).

Because the foams have the limited applications, the following attention was concentrated on polymers and composites, in which the internal structures can be formed that transform the composites into the auxetics. One of promising directions consists in creating the cushioning materials (body armours, knee-caps, packing materials and so on). The gaskets and compactions made of auxetics seem to be perspective. Some rocks and minerals as well as the specially manufactured metamaterials are related now to auxetics.

Thus, the presented short information on auxetics shows that their existence changes the postulates of classical theory of elasticity and some facts from postulates of this theory should be reconsidered (refined).

First, their definition is based on the secondary fact - negativity of the Poisson ratio, which corresponds to the model of linearly elastic body. The primary fact consists in observation in the standard for mechanics of materials (which does not depend on the model of deformation) experiment of longitudinal tension of a prism that the transverse deformation of prism is positive (a material as if swells) in contrast to the classical materials, where it is negative.

Second, the classical comments of elastic constants should be refined according to new knowledge of modern materials, namely, the auxetic materials.

3. Elastic constants in the classical theory of elasticity

Consider first the linear isotropic model. In this case, the deformation of elastic material is described by the Young modulus E, two Lame modulus λ and μ (shear modulus), compression modulus k and Poisson ratio ν. Two only of five physical constants above are independent. One only modulus λ has not a physical sense. The physical sense is explained on so called uniform (universal) deformations. The classical theory of elasticity knows a few uniform stress states: unilateral tension, omnilateral compression, pure shear. Analysis of such states belongs to the foundations of the theory of elasticity and its results are considered as ultimate and irrevocable.

The best illustration of such analysis can be found in the classical Love's book [13]. First write according to [13] the standard presentation of the Hooke law through the Lame moduli λ, μ

$$X_x = \lambda\Delta + 2\mu\varepsilon_{xx}, \quad Y_y = \lambda\Delta + 2\mu\varepsilon_{yy}, \quad Z_z = \lambda\Delta + 2\mu\varepsilon_{zz}, \quad X_y = 2\mu\varepsilon_{xy}, Y_z = 2\mu\varepsilon_{yz}, Z_x = 2\mu\varepsilon_{zx}. \quad (2)$$

and the standard notation of dilatation $\Delta = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$. Here the adopted at that time notations are saved.

Then the internal energy (the accumulated owing deformation energy) of the linearly elastic isotropic body W can be represented in the form

$$W = \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})^2 + 2\mu(\varepsilon_{xx}^2 + \varepsilon_{yy}^2 + \varepsilon_{zz}^2) + \mu(\varepsilon_{xy}^2 + \varepsilon_{yz}^2 + \varepsilon_{xz}^2). \quad (3)$$

This is the initial point for the Love's analysis of physical constants.

Let us start now with the classical procedure of introducing the Young modulus and Poisson ratio. Toward this end, the cylinder or prism of any shape is considered. It is assumed also that the axis is chosen in direction Ox and the prism is stretched at the ends by an uniform tension T. Because the lateral surface of prism is assumed to be free of stresses, then the stress state of prism is uniform and is only characterized by the stress $X_x = T$ (other stresses are zero ones). In this case, the Hooke law becomes simpler — it includes three components of the strain tensor

$$T = \lambda\Delta + 2\mu\varepsilon_{xx}, \quad 0 = \lambda\Delta + 2\mu\varepsilon_{yy}, \quad 0 = \lambda\Delta + 2\mu\varepsilon_{zz}, \tag{4}$$

This system is the inhomogeneous system of three linear algebraic equations with three unknown variables $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}$ and can be analyzed by different ways. A following analysis of equations (4) consists of two steps. First, the dilatation is obtained by adding the all three equations (4)

$$T = (3\lambda + 2\mu)\Delta \rightarrow \Delta = T/(3\lambda + 2\mu). \tag{5}$$

Second, the substitution of expression for dilatation (5) into the first equality (4) gives relation between tension T and strain ε_{xx}

$$T = \frac{\mu(3\lambda + 2\mu)}{(\lambda + \mu)}\varepsilon_{xx}. \tag{6}$$

The expression (6) represents the elementary law $T = E\varepsilon_{xx}$ of link between tension and deformation of prism, in which the Young modulus is used. Juxtaposition of relation (6) with this law permits to write the classical expression of the Young modulus through the Lame moduli

$$E = \frac{\mu(3\lambda + 2\mu)}{(\lambda + \mu)}. \tag{7}$$

The substitution of expression for dilatation into the second and third equalities (5) gives relations

$$-\varepsilon_{yy} = -\varepsilon_{zz} = \frac{\lambda}{2(\lambda + \mu)}\varepsilon_{xx}. \tag{8}$$

Formula (8) expresses the mentioned above classical Poisson law on the transverse compression under the longitudinal extension and permits to introduce the Poisson ratio

$$\sigma = \frac{-\varepsilon_{yy}}{\varepsilon_{xx}} = \frac{-\varepsilon_{zz}}{\varepsilon_{xx}} = \frac{\lambda}{2(\lambda + \mu)}. \tag{9}$$

Note that formula (9) corresponds to the Love's representation of the Poisson ratio. This elastic constant can be represented through any two other elastic moduli.

Note also that two experimental approaches to determine the value of Poisson ratio for concrete material are used at present time [26, subsections 2.18, 3.27, 3.28]. The first approach is older one. It is based on experimental determination of Young, shear, and compression moduli and subsequent calculation of the Poisson ratio by formulas $\sigma = (E/2\mu) - 1$, $\sigma = (1/2)[(E/3k) - 1]$. Here the problem of exactness of calculation arises. Let us cite the Bell's book [26, subsection 3.28]: "Remind of the Gruneisen's conclusion that the errors of $\pm 1\%$ in values E and μ result in the error of 10% in the value of Poisson ratio". Therefore, the second approach seems to be more preferable. It is associated with Kirchhoff's experiments (1859), in which the Poisson ratio is determined from the direct experiment on simultaneous bending and torsion.

Let us recall that the primary phenomenon in determination of the Poisson ratio is the contraction of a sample (transverse deformation of a sample) under its elongation (its longitudinal deformation).

Consider now the second universal (uniform) deformation — the uniform compression — according to Love's book [13]. In this case, the body of arbitrary shape is considered, to all surface points of which the constant pressure $(-p)$ is applied. Then the uniform stress state arises in this body, which is characterized by stresses $X_x = Y_y = Z_z = -p, \quad X_y = Y_z = Z_x = 0$. The Hooke law can be written as follows

$$-p = \lambda\Delta + 2\mu\varepsilon_{xx}, \quad -p = \lambda\Delta + 2\mu\varepsilon_{yy}, \quad -p = \lambda\Delta + 2\mu\varepsilon_{zz}, \quad \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{zx} = 0. \tag{10}$$

If to add all three equations (10) $-3p = (3\lambda + 2\mu)(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) \rightarrow -3p = (3\lambda + 2\mu)\Delta$, then the modulus of compression can be defined

$$k = \lambda + (2/3)\mu. \tag{11}$$

The classical Love's reasoning, which are repeated in the most part of books on the linear theory of elasticity, is based on representation of moduli λ, μ, k through moduli E, σ

$$\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}, \quad \mu = \frac{E}{2(1+\sigma)}, \quad k = \frac{E}{3(1-2\sigma)}. \tag{12}$$

The formulas (12) are commented in [13, page 104] as follows: "If σ were $> (1/2)$, μ would be negative, or the material expand under pressure. If σ were < -1 , μ would be negative, and the function W would not be a positive quadratic function. We may show that this would also be the case if k were negative. Negative values for σ are not excluded by the condition of stability, but such values have not been found for any isotropic material".

Because the comments of negativity of Poisson ratio is found in the books on theory of elasticity very seldom, therefore two sentences from the Lurie's book [2, page 117] are worthy to be cited: "A tension of the rod with negative ν (but the more than -1) would be accompanied by increasing of transverse sizes. Energetically, an existence of such elastic materials is not excluded." "In hypothetic material with $\nu < -1$, the hydrostatic compression of the cube would accompanied by increasing its volume".

It should be noted that not all authors of books on the linear isotropic theory of elasticity discuss the restrictions on changing the Poisson ratio (for example, Timoshenko [14], Germain [17], Nowacki [15] or Hahn [18] does not made this).

But in some books the discussion is presented and all authors start with one and the same postulate: in the procedure of restrictions in changing the Poisson ratio, the primary requirement is a positiveness of internal energy W. The representation of energy can be different for different elastic moduli. For example, Leibensohn [12], Love [13], Lurie [2] choose the pair λ, μ and use the representation (1). Landau and Lifshits [16] use the pair k, μ. In all the cases, W has a form of quadratic function with coefficients composed of elastic moduli.

Thus, in most cases the expression (3) is analyzed. The basic in this analysis assumption is the assumption of positiveness of Lame moduli

$$\lambda > 0, \quad \mu > 0, \tag{13}$$

which is considered as the sufficient and being in line with experimental observations condition.

And further the formulas (12) are considered, in which the Young modulus is assumed positive without controversy. Then positiveness of expressions $1 + \sigma > 0$, $1 - 2\sigma > 0$ provides validity of formula (13), from which the well-known restriction on the Poisson ratio (1) follows.

Let us recall that all the elastic constants are always positive in the classical linear isotropic theory of elasticity. The obvious contradiction between assumption of negativity of the Poisson ratio and the primary statement on positivity of Lame moduli (11) in condition when the Poisson ratio is defined by formula (9) is commented in the classical theory of elasticity anybody. To all appearance, this situation is occurred owing to incredibility of negative values if only one elastic moduli λ, μ, E, k.

4. Elastic moduli in the modern theory of elasticity

The starting point in this section is that nowadays the experimental (within the framework of mechanics of materials) and theoretical (within the framework of molecular physics) verifications are reported that the elastic materials with negative Poisson ratio exist [19,20,23,25,27]. These new materials can be linear and nonlinear as well as they can have different levels of symmetry. Therefore further the term "modern theory of elasticity" is used in contrast to the classical theory of elasticity, where the negative values of Poisson ratio were not discussed. Below, three cases of auxetic materials will be considered: isotropic linear, transversally isotropic linear, and isotropic nonlinear.

4.1. Case of isotropic linear auxetic materials

Let us save the initial postulate of the classical theory of elasticity: in all the analysis of elastic constants, the primary requirement is the requirement of positivity of internal energy

$$W = \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})^2 + 2\mu(\varepsilon_{xx}^2 + \varepsilon_{yy}^2 + \varepsilon_{zz}^2) + \mu(\varepsilon_{xy}^2 + \varepsilon_{yz}^2 + \varepsilon_{xz}^2) > 0. \tag{14}$$

But further the sufficient (and not necessary) condition of positivity of (14) internal energy W, when the positivity of Lame moduli λ, μ is assumed, is rejected and the general condition of positivity of (14), when one of moduli can be negative, is supposed. This step is just caused by intention to include the auxetic materials into the family of elastic materials.

It seems to be appropriate to consider first the Lame modulus μ . It has the physical sense of the shear modulus and until now the facts of observation its negativity (that is, observation of a shear in direction opposite to direction of shear force) are not reported. Then we can agree to its positivity

$$\mu > 0 \tag{15}$$

This condition can be also substantiated theoretically on the base of analysis of universal (uniform) state of simple shear. To describe the simple shear, the coordinate plane (for example, x_1Ox_2) should be chosen and only one non-zero component of the displacement gradient $u_{1,2}$ should be given. This can be commented geometrically as deformation of elementary rectangle $ABCD$ with sides dx_1, dx_2 parallel to the coordinate axes into the parallelogram $AB'C'D$, which results by the longitudinal shift of the rectangle side BC . Then shear angle $\measuredangle BAB' = \gamma$ is linked with the component $u_{1,2}$ in the next way $u_{1,2} = \tan\gamma = \tau$, $\varepsilon_{12} = (1/2)\tau$. The Hooke law becomes the simplest form $\sigma_{12} = 2\mu\varepsilon_{12}$ and the corresponding representation of internal energy is as follows: $W = (1/2)\mu\tau^2$. The positivity of shear modulus (15) follows from positivity of energy W. But in this case, the classic expression for Poisson ratio (9) shows that negativity of the Poisson ratio σ and positivity of the shear modulus μ should be accommodated by negativity of the Lame modulus λ .

This permits to formulate the first two complements to the modern theory of elasticity.

Complement 4.1.1. In the case of modern isotropic linear theory of elasticity, the Lame modulus λ must be negative, if the Poisson ratio σ is assumed possible negative.

Complement 4.1.2. In the case of modern isotropic linear theory of elasticity, the assumption that the Poisson ratio σ is possible negative and the shear modulus μ is positive is resulted in that according to definition (9) the negative Lame modulus λ can not exceed by absolute value the shear modulus

$$|\lambda| < \mu \tag{16}$$

Let us recall that the primary definition of the Poisson ratio (9) through the Lame moduli λ, μ is found from the problem on unilateral tension. In this case, the internal energy has the form

$$\begin{aligned} W &= \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})^2 + 2\mu(\varepsilon_{xx}^2 + \varepsilon_{yy}^2 + \varepsilon_{zz}^2) = \\ &= \lambda(\varepsilon_{xx} + 2\sigma\varepsilon_{xx})^2 + 2\mu(\varepsilon_{xx}^2 + 2(\sigma\varepsilon_{xx})^2) = [\lambda(1+2\sigma)^2 + 2\mu(1+2(\sigma)^2)]\varepsilon_{xx}^2 > 0. \end{aligned} \tag{17}$$

Thus, the expression in the brackets should be positive

$$\lambda + (1+2\sigma)^2 + 2\mu(1+2(\sigma)^2) > 0 \to \lambda + 2(1+2(\sigma)^2)/((1+2\sigma)^2\mu > 0. \quad (18)$$

The formula (18) permits to formulate some new complement.

Complement 4.1.3. In the case of modern isotropic linear theory of elasticity, if the Poisson ratio σ is assumed to be possible negative and the shear modulus μ is positive, then the condition of positivity of internal energy (15) admits arbitrary negative values of the Poisson ratio (because the coefficient ahead of μ is always positive). The case $1+2\sigma = 0 \to \sigma = -(1/2)$ is the peculiar one — the value of modulus is practically not restricted at its neighborhood.

The Lame modulus λ is already restricted from below according to (16), but also the additional condition (18) exists

$$|\lambda| < 2(1+2(\sigma)^2)/((1+2\sigma)^2)\mu. \quad (19)$$

The condition (19) is less strong: the coefficient ahead of μ exceeds 1 for all negative σ , whereas in condition (16) the coefficient ahead of μ is equal to 1). Therefore, the condition (16) remains.

Let us turn to formula (11), which expresses the compression modulus k through the Lame moduli λ, μ. It follows from (11) that the modulus k will be negative if only the negative Lame modulus λ exceeds $(2/3)\mu$ by absolute value

$$k = \lambda + (2/3)\mu < 0 \to |\lambda| > (2/3)\mu. \quad (20)$$

Comparison with restrictions (16) and (19) on the absolute values of negative Lame modulus shows that, in the case of negative values of Poisson ratio σ, restriction (20) does not conflict with (16) and (19).

Complement 4.1.4. In the case of modern isotropic linear theory of elasticity, if the Poisson ratio σ is assumed to be possible negative and the shear modulus μ is positive, then the compression modulus k can be negative.

Consider together the formula (7), which expresses the dependence of the Young modulus E on the Lame moduli λ, μ, and formula (12), which expresses the dependence of compression modulus k on the Young modulus E and Poisson ratio σ. Then the conclusion can be formulate that the Young modulus also can be negative. The situation becomes clearer if the moduli λ, E and k are written through μ (positive) and σ (negative)

$$\lambda = \frac{2\sigma}{(1-2\sigma)}\mu, \quad E = 2(1+\sigma)\mu, \quad k = \frac{2}{3}\frac{1+\sigma}{(1-2\sigma)}\mu. \quad (21)$$

This fact can be formulated as new complement.

Complement 4.1.5. In the case of modern isotropic linear theory of elasticity, if the Poisson ratio σ is assumed to be possible negative and the shear modulus μ is positive, then the Young modulus E can be negative.

The shown above analysis of the case of isotropic linear auxetic materials permits to formulate some statements.

Statement 4.1.1. Seemingly, the auxetics should be defined by the primary physical phenomenon of positivity of transverse deformation of a prism, which is observed in the standard in mechanics of materials experiment of longitudinal tension of a prism. In this case, the auxetics will be associated not only with the isotropic elastic materials.

Statement 4.1.2. In the case of isotropic auxetic materials, the Lame modulus λ is always negative. The Young E and compression k moduli are negative, when the negative Poisson ratio is less than -1.

Statement 4.1.3. The classical restrictions of positivity of the elastic moduli in the isotropic theory of elasticity should be refined and not used in the modern theory of elasticity, where the most part of elastic moduli can be negative. In this case, the restrictions (16) and (20) for the Lame modulus λ can be used.

Statement 4.1.4. When the problems of the modern linear isotropic theory of elasticity being studied for auxetic materials, then at least two elastic moduli for these materials should be determined from the direct experiments (unilateral tension, omnilateral compression, simple shear, torsion).

4.2. Case of transversely isotropic linear auxetic materials

Let us start with the classical representation of constitutive relations for the transversally isotropic linearly elastic materials [6], which is described by five independent elastic moduli

$$X_x = C_{xx}\varepsilon_{xx} + C_{xy}\varepsilon_{yy} + C_{xz}\varepsilon_{zz},\ Y_y = C_{xy}\varepsilon_{xx} + C_{xx}\varepsilon_{yy} + C_{xz}\varepsilon_{zz},\ Z_z = C_{xz}\varepsilon_{xx} + C_{xz}\varepsilon_{yy} + C_{zz}\varepsilon_{zz}),$$
$$Y_z = (1/2)C_{xzxz}\varepsilon_{yz}, \quad Z_y = (1/2)C_{xzxz}\varepsilon_{xz}, \quad X_y = (1/2)(C_{xx} - C_{xy})\varepsilon_{yz}. \tag{22}$$

Here, the main axis and the plane of isotropy are chosen as Oz and xOy, respectively, and C_{xx}, C_{zz}, C_{xy}, C_{xz}, C_{xzxz} are the elastic constants.

Then the internal energy can be written in the form

$$2W = C_{xx}(\varepsilon_{xx})^2 + C_{xx}(\varepsilon_{yy})^2 + C_{zz}(\varepsilon_{zz})^2 + 2C_{xy}\varepsilon_{xx}\varepsilon_{yy} + 2C_{xz}\varepsilon_{xx}\varepsilon_{zz} + 2C_{yz}\varepsilon_{yy}\varepsilon_{zz} +$$
$$+ (1/2)(C_{xx} - C_{xy})(\varepsilon_{xy})^2 + C_{xzxz}(\varepsilon_{xz})^2 + C_{xzxz}(\varepsilon_{yz})^2. \tag{23}$$

Introduce also the technical elastic constants, which have the physical interpretation: $E_z = C_{zz} - 2(C_{xz})^2/(C_{xx} + C_{xy})$ (the longitudinal modulus of elasticity, the Young modulus in direction of symmetry axis), $E_x = E_y = [(C_{xx} - C_{xy})(C_{xx}C_{zz} + C_{xy}C_{zz}) - 2(C_{xz})^2]/(C_{xx}C_{zz} - (C_{xz})^2)$ (the transverse modulus of elasticity, the Young modulus in the plane of isotropy), $G_{xz} = C_{xzxz}$(the longitudinal shear modulus, the shear modulus in direction of symmetry axis), $G_{xy} = (1/2)(C_{xx} - C_{xy})$ (the transverse shear modulus, the shear modulus in the plane of isotropy), $\nu_{xy} = (E_z/2G_{xy}) - 1$ (the Poisson ratio characterizing the strain in the isotropy plane under tension in this plane), $\nu_{xz} = [C_{xz}/(C_{xx} + C_{xy})]$ (the Poisson ratio characterizing the strain in the isotropy plane under tension perpendicular to this plane), $\nu_{zx} = [((C_{xz})^2 - C_{xy}C_{zz})/((C_{xz})^2 - C_{xx}C_{zz})]$ (the Poisson ratio characterizing the strain in direction of symmetry axis under tension in the isotropy plane).

Note that the Poisson ratio ν_{xy} is introduced like the case of isotropic auxetic material, because it characterizes the increasing (decreasing) of the cross-section in the isotropy plane. This fact permits to formulate new statement.

Statement 4.2.1. In the case of auxetic transversally isotropic materials, the Young modulus E_z is negative, when the negative Poisson ratio ν_{xy} is less than -1: $\nu_{xy} < -1$. Also, the formula $\nu_{xy} = (E_z/2G_{xy}) - 1$ shows that, in this case of negativity of the Poisson ratio ν_{xy}, the Young modulus E_z will not exceed the doubled shear modulus G_{xy}.

The constitutive equations (22) can be also represented through the technical elastic constants

$$\begin{aligned}
\varepsilon_{xx} &= (1/E_x)X_x - (\nu_{xy}/E_x)Y_y - (\nu_{xz}/E_z)Z_z,\\
\varepsilon_{yy} &= -(\nu_{xy}/E_x)X_x + (1/E_x)Y_y - (\nu_{xz}/E_z)Z_z,\\
\varepsilon_{zz} &= -(\nu_{xz}/E_x)X_x - (\nu_{xy}/E_x)Y_y + (1/E_z)Z_z,\\
\varepsilon_{xy} &= (2(1+\nu_{xy})/E_x)X_x - (\nu_{xy}/E_x)Y_y - (\nu_{xz}/E_z)Z_z,
\end{aligned} \tag{24}$$

Consider now the first uniform stress state, when the prism is stretched at the ends by an uniform tension $Z_z = T$ in direction Oz. Then the constitutive equations become simpler

$$0 = C_{xx}\varepsilon_{xx} + C_{xy}\varepsilon_{yy} + C_{xz}\varepsilon_{zz},\ 0 = C_{xy}\varepsilon_{xx} + C_{xx}\varepsilon_{yy} + C_{xz}\varepsilon_{zz},\ Z_z = C_{xz}\varepsilon_{xx} + C_{xz}\varepsilon_{yy} + C_{zz}\varepsilon_{zz} \tag{25}$$

or

$$\varepsilon_{xx} = -(\nu_{xz}/E_z)Z_z, \quad \varepsilon_{yy} = -(\nu_{xz}/E_z)Z_z, \quad \varepsilon_{zz} = (1/E_z)Z_z \tag{26}$$

and the internal energy can be written in the form

$$W = (1/2)(Z_z)^2(1/E_z)^2[2(C_{xx} + C_{xy})(\nu_{xz})^2 - 4C_{xz}\nu_{xz} + C_{zz}]. \tag{27}$$

Let us save the initial postulate of the classical theory of elasticity (14) of positivity of W. Then the condition $2(C_{xx} + C_{xy})(\nu_{xz})^2 - 4C_{xz}\nu_{xz} + C_{zz} > 0$ must be fulfilled. It can be simplified to

$$(C_{xx} + C_{xy})\nu_{xz}[-\nu_{zx} + (C_{zz}/2C_{xz})] > 0 \tag{28}$$

Suppose now that the Poisson ratio ν_{xz} is negative, then one of blocks $(C_{xx} + C_{xy})$, $(C_{zz}/2C_{xz})$ must be negative, too. This means that, at least, one of elastic constants C_{xx}, C_{xy}, C_{zz}, $2C_{xz}$ must be negative. Therefore, the next statement can be formulated.

Statement 4.2.2. For the transversely isotropic auxetic materials, the negativity of the Poisson ratio ν_{xz} is accompanied by negativity of some other elastic constants.

4.3. Case of isotropic nonlinear auxetic materials

Consider now the most known variant of the nonlinear theory of elasticity, which is based on the Murnaghan elastic potential [1–5]

$$W = (1/2)\lambda(\varepsilon_{mm})^2 + \mu(\varepsilon_{ik})^2 + (1/3)A\varepsilon_{ik}\varepsilon_{im}\varepsilon_{km} + B(\varepsilon_{mm})^2\varepsilon_{mm} + (1/3)C(\varepsilon_{mm})^3. \tag{29}$$

It includes five elastic constants λ, μ, A, B, C and therefore is called sometimes the five-constant potential.

The main goal of next analysis is to look at the simplest uniform stress state within the proposed nonlinear statement — the state of pure shear. Note that this state is considered in subsection 4.1 within the linear statement. The following nonlinear analysis needs, like the linear one, the constitutive equation for the tangential stress σ_{12} and corresponding to the pure stress state representation of internal energy W. As it is well-known [1–5], σ_{12} and W are linked by formula $\sigma_{12} = (\partial W/\partial\varepsilon_{12})$. Let us start with the energy and write only the terms, which contain ε_{12}

$$\begin{aligned} W = {} & \ldots + \mu(\varepsilon_{ik})^2 + (1/3)A\varepsilon_{ik}\varepsilon_{im}\varepsilon_{km} + B(\varepsilon_{mm})^2\varepsilon_{mm} + \ldots = \\ & = \mu[(\varepsilon_{11})^2 + (\varepsilon_{22})^2 + (\varepsilon_{33})^2 + 2(\varepsilon_{12})^2 + 2(\varepsilon_{13})^2 + 2(\varepsilon_{23})^2] + \\ & \quad + (1/3)A[(\varepsilon_{11}\varepsilon_{11} + \varepsilon_{21}\varepsilon_{21} + \varepsilon_{31}\varepsilon_{31})\varepsilon_{11} + (\varepsilon_{12}\varepsilon_{12} + \varepsilon_{22}\varepsilon_{22} + \varepsilon_{32}\varepsilon_{32})\varepsilon_{22} + \\ & \quad + (\varepsilon_{13}\varepsilon_{13} + \varepsilon_{23}\varepsilon_{23} + \varepsilon_{33}\varepsilon_{33})\varepsilon_{33} + 2(\varepsilon_{11}\varepsilon_{12} + \varepsilon_{21}\varepsilon_{22} + \varepsilon_{31}\varepsilon_{32})\varepsilon_{12} + \\ & \quad + 2(\varepsilon_{11}\varepsilon_{13} + \varepsilon_{21}\varepsilon_{23} + \varepsilon_{31}\varepsilon_{33})\varepsilon_{13} + 2(\varepsilon_{12}\varepsilon_{13} + \varepsilon_{21}\varepsilon_{23} + \varepsilon_{32}\varepsilon_{33})\varepsilon_{13}] + \\ & \quad + B[(\varepsilon_{11})^2 + (\varepsilon_{22})^2 + (\varepsilon_{33})^2 + 2(\varepsilon_{12})^2 + 2(\varepsilon_{13})^2 + 2(\varepsilon_{23})^2](\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}). \end{aligned} \tag{30}$$

Then

$$\sigma_{12} = 2\mu\varepsilon_{12} + 2A(\varepsilon_{11}\varepsilon_{12} + \varepsilon_{22}\varepsilon_{12} + \varepsilon_{13}\varepsilon_{23}) + 2B(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})\varepsilon_{12}. \tag{31}$$

This expression for the tangential stress in the nonlinear problem on pure shear looks very simple and it permits to formulate a few quite simple statements.

Statement 4.3.1. It consists of linear and nonlinear parts. The linear part corresponds to the linear theory of elasticity and includes one only component of the strain tensor. The nonlinear part includes all six components of the strain tensor.

Statement 4.3.2. In the nonlinear case, the physical constant μ has not the physical sense of the shear modulus, because the characterizing the state of pure shear tangential stress is described by three elastic constants μ, A, B. More generally, any elastic constant of the Murnaghan elastic potential has the physical sense.

Statement 4.3.3. But this means that the Poisson coefficients can not be used in the nonlinear theory of auxetic materials. These materials should be defined in the nonlinear approaches by the primary physical effect — observation in the standard for mechanics of materials (which does not depend on the model of deformation) experiment of longitudinal tension of a prism that the transverse deformation of prism is positive (a material as if swells) in contrast to the classical materials, where it is negative.

5. Conclusions

The auxetic materials are considered from the point of view of correspondence to the classical theory of elasticity. It is shown that some classical postulates relative to the elastic constants should be refined. Three cases of description of auxetic materials — by the model of linear elastic isotropic body, by the model of linear elastic transversally isotropic body, by the model of nonlinear elastic isotropic body (Murnaghan potential) — are analyzed shortly. The initial assumption on positivity of internal energy of deformation is saved and then the uniform stress states (unilateral tension, omnilateral compression, pure shear) are used to analyze the elastic constants. This allows to describe the new mechanical effects: expansion of the standard sample-rod-prism under unilateral tension and expansion of the standard sample-cube under hydrostatic compression as well as an existence of the arbitrary negative values of Poisson ratio ν, what is accompanied by the negative values of the Lame λ, Young E and compression k moduli, for the linear isotropic case and some elastic constants in the linear transversely isotropic case. The case of nonlinear description shows that the auxetic materials should be defined by the primary physical effect - observation in the standard for mechanics of materials experiment of longitudinal tension of a prism that the transverse deformation of prism is positive (a material as if swells) in contrast to the classical materials, where it is negative.

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До моделювання ауксетичних матеріалів: деякі фундаментальні аспекти

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Розглянуто ауксетичні матеріали з точки зору відповідності класичній теорії пружності. Показано, що деякі класичні постулати щодо пружних констант повинні бути уточнені. Проаналізовано коротко три випадки опису ауксетичних матеріалів — за допомогою моделі лінійного пружного ізотропного тіла, за допомогою моделі лінійного пружного трансверсально ізотропного тіла, за допомогою моделі нелінійного пружного ізотропного тіла (потенціал Мернагана). Збережено початкове припущення про додатність внутрішньої енергії деформування і далі використано однорідні напружені стани (одновісний розтяг, всесторонній стиск, чистий зсув) для аналізу пружних констант. Це дозволяє описати нові механічні ефекти: розширення стандартного зразка-стержня-призми при одновісному розтязі і розширення стандартного зразка-куба при гідростатичному стиску, а також існування довільних від'ємних значень коефіцієнтів Пуассона σ, яке супроводжується від'ємними значеннями модулів Ляме λ, Юнга E і всестороннього стиску k у випадку лінійного ізотропного тіла і деяких пружних констант у випадку лінійного трансверсально ізотропного тіла. Аналіз випадку нелінійного ізотропного тіла підтвердив попередні спостереження, що ауксетичні матеріали повинні означатися за первинним фізичним явищем — спостереженням у стандартному для механіки матеріалів експерименті про одновісний розтяг призми, що поперечна деформація призми є додатною (матеріал наче розбухає) на відміну від класичних матеріалів, де ця деформація є відємною.

Ключові слова: *ауксетичний матеріал, ізотропний матеріал, анізотропний матеріал,від'ємні значення пружних модулів, обмеження на пружні модулі*

2000 MSC: 74B99

УДК: 539.3